WAVE MOTIONS IN THE OCEAN

presented to

Myrl C. Hendershott

from

David C. Chapman and Paola Malanotte-Rizzoli

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Preface

When I volunteered to teach the MIT/WHOI Joint Program core course on "Wave Motions in the Ocean and Atmosphere" in Spring 1989, I naturally turned for guidance to the notes I had acquired from a similar course taken while a student at Scripps Institution of Oceanography. In an attempt to broaden the scope of the course, I borrowed a set of notes from Paola Malanotte-Rizzoli who taught the MIT/WHOI core course from 1983-1985. It didn't take long to recognize that Paola's notes were nearly identical to mine because she had also based hers on the waves course she had taken at Scripps. In both cases, the Scripps course was taught by our former advisor Myrl Hendershott, which means that at least two generations of Physical Oceanography students have learned the "Hendershott view" of waves. Considering the seemingly timeless nature of the concepts presented in Myrl's course as well as the profound influence Myrl has had on Paola and myself through both his teaching and his advising, we decided to compile these notes into a form which could be distributed to students and, at the same time, serve as a tribute to Myrl. Thus, with the exception of some minor modifications, additions and deletions that Paola and I have made, the notes contained herein are those developed by Myrl for his course. We hope that these notes will be as clear and as useful to future readers as they have been to us.

Woods Hole
1989

David C. Chapman

These notes have been collected and assembled in different ways over the years by two people successively, Paola Malanotte-Rizzoli and Dave Chapman. The present and chronologically latest version has been put together by Dave and constitutes the bulk of the waves course he taught in Spring 1989. When I taught the course during the years 1983-85, the chapter on acoustic waves was absent. I had instead a section on the Garrett and Munk spectrum and a chapter on nonlinear wave interactions. These differences reflect the different years in which Dave and I took the waves course at Scripps Institution of Oceanography from our former advisor Professor Myrl C. Hendershott and the modifications that Myrl had made in his course in successive years. Thus the inspirational source or, rather, the actual bulk of these notes *is* the waves course taught by Myrl at Scripps.

Myrl Hendershott has been at W.H.O.I. this summer as Principal Lecturer of the GFD Summer School on Ocean Circulation. This opportunity, plus Dave Chapman's diligence and patience in typing the notes on his word processor together with formulas and equations (the latter were handwritten in my own set of notes), has motivated us to produce this report as an homage to Myrl. Without him, we would both have had a much harder and more time-consuming role in putting together a decent course on waves. More importantly, Myrl is in many ways responsible for whatever success we have had in the field of Oceanography.

I must add here a personal note. Hearing Myrl again as a teacher this summer after so many years, I have realized how much he has influenced my way of thinking and teaching. On the not-so-positive side (I will *not* say negative):

- like him, I "scribble" a lot on the blackboard.
- like him, I erase with my left hand what I have just written with my right hand.
- like him, I put ℓ (x wavenumber) before k (y wavenumber)

As the letters j, k, x, y, w do not exist in the Italian alphabet, k coming before or after ℓ was supremely unimportant to me. On the positive side, Myrl was absolutely the best teacher I had in the various courses I took at Scripps. His lectures were always interesting, imaginative and full of physical insight. Looking back, I realize that a great deal of the important oceanographic concepts and ideas I learned over the years go back to my long association with Myrl as teacher, advisor, colleague and, last but not least, dear friend. I hope I absorbed from him some of the positive qualities too.

Woods Hole
1989

Paola Malanotte-Rizzoli

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Chapter 1

Basic concepts

Waves are not easy to define. Whitham (1974) defines a wave as "a recognizable signal that is transferred from one part of a medium to another with recognizable velocity of propagation". This is a very broad definition and encompasses an enormous range of dynamical systems as well as physical processes. That is, waves can occur in many different media and take on many different forms. We often think of waves as simple sinusoidal undulations of some substance, but this view is too restricted and often not very useful.

In this course, we will consider a number of different types of waves and wave motions in the ocean and in the atmosphere. They will be found to occur at many different time and space scales. In general, wave-like fluctuations of flow fields are *not* exact solutions of the continuum formulation of momentum and mass conservation and the laws of thermodynamics. However, they often represent good *approximate* solutions of those equations.

Therefore, the first step in discussing wave motion is the appropriate simplification of the field equations to obtain a set whose solutions are waves. In most of what we do, this involves *linearizing* the field equations about some basic state of rest or of quasi-steady motion. That is, products of any dependent variables in the equations are typically assumed to be small in relation to the other terms. It usually proves possible, by this device, to obtain waves as solutions of the linearized equations.

Because the equations are linear, we are entitled to superpose solutions of the equations in order to find solutions to more general initial and boundary value problems. This is one of the real beauties of linear wave theory. We will spend most of our time studying such linear waves and their properties before relaxing the linearization condition which precludes nonlinear interactions.

As we will see, there are many different waves with quite different characteristics which can exist within the framework of rotating fluid systems such as the ocean and the atmosphere. In order to proceed, certain concepts and approaches which are common to most studies of linear waves should be understood first. Some of these are presented next.

1.1 Plane waves

The basic state of rest or quasi-steady flow about which the waves are linear perturbations defines the medium through which the waves propagate. If we *assume* that the medium is homogeneous in space and time (even if it strictly is not), then possible solutions often have the form of a *plane wave*:

$$\phi(\vec{x}, t) = \Re \; A e^{i(\vec{k}\cdot\vec{x} - \sigma t)}$$

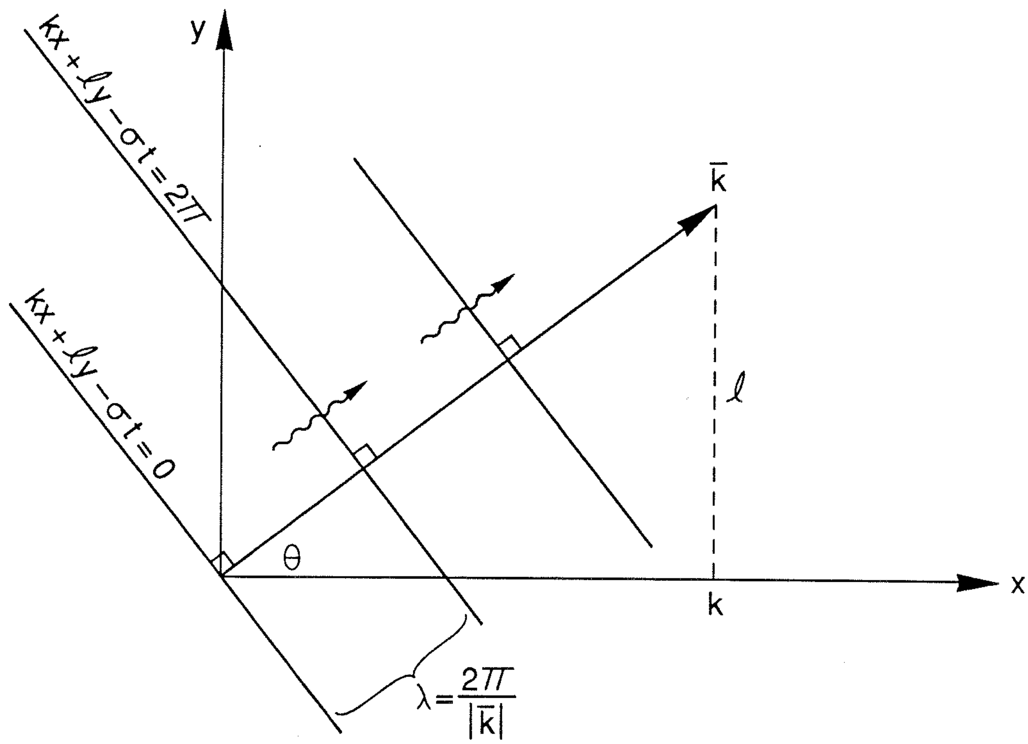
where $\phi(\vec{x}, t)$ are the dependent variables (i.e., velocity $\vec{u}$, pressure p, density ρ, etc.), A is the amplitude, $\vec{k} = (k, \ell, m)$ is the wavenumber, σ is the radian frequency, and $\Re$ means that we take the real part of the expression. Customary auxiliary definitions are $\lambda = 2\pi/|\vec{k}|$ = wavelength, $f = \sigma/2\pi$ = frequency, $T = 2\pi/\sigma = 1/f$ = period.

Since A is complex, it carries with it not only amplitude but also phase information. We could, of course, write

$$\phi(\vec{x}, t) = |A| \cos(\vec{k} \cdot \vec{x} - \sigma t + \tan^{-1} \frac{\Im A}{\Re A})$$

where $\Im$ refers to the imaginary part of the expression. However, it is often much more convenient to work with the complex form of all variables and to take the real parts only at the very end. This is always possible because we have linearized the field equations.

The convention $e^{i(\vec{k}\cdot\vec{x}-\sigma t)}$ is preferable to the convention $e^{i(\vec{k}\cdot\vec{x}+\sigma t)}$ because, in the first case, wave 'crests and troughs' move in the direction of $\vec{k}$ when $\sigma > 0$. This can be seen by examining the *phase* of the wave, namely $\vec{k} \cdot \vec{x} - \sigma t$. Surfaces of constant phase, $\vec{k} \cdot \vec{x} - \sigma t = \Phi_0$, are planes normal to $\vec{k}$ and moving outward along $\vec{k}$ as t increases (for $\sigma > 0$). In two dimensions we have



The speed at which phase planes move along $\vec{k}$ is the *phase speed*

$$c = \sigma/|\vec{k}| = \lambda/T$$

It is directed along $\vec{k}$. Note that the speed of phase plane intersection with the x-axis is not $c\cos\theta$ but rather is

$$\frac{c}{\cos\theta} = (\frac{\sigma}{|\vec{k}|})/(\frac{k}{|\vec{k}|}) = \sigma/k$$

which can be considerably faster than c. In fact, as $\theta \to \pi/2$, the phase speed in the x-direction approaches infinity!

The form $Ae^{i(\vec{k}\cdot\vec{x}-\sigma t)}$ is called a 'travelling plane wave'. The superposition of oppositely travelling plane waves

$$Ae^{i(\vec{k}\cdot\vec{x}-\sigma t)} + Ae^{i(-\vec{k}\cdot\vec{x}-\sigma t)} = 2Ae^{-i\sigma t}\cos(\vec{k}\cdot\vec{x})$$

is called a *standing wave* because the crests and troughs do not propagate with time. It is not always possible to construct such a superposition because oppositely travelling plane waves are not always possible and, even when possible, may have different wavenumbers.

1.2 The dispersion relation

All of the foregoing is kinematics, true for any given $\sigma, \vec{k}$ with no physics. The physics are contained in the *dispersion relation*

$$\sigma = \Omega(\vec{k})$$

which is obtained by requiring the plane waves to be solutions of the linearized, dissipationless equations of motion. The following table contains some examples of wave equations (all of which we will encounter later) with their respective dispersion relations.

Linearized Equation	Plane wave	Dispersion Relation
a) $\phi_t + c_0\phi_x = 0$	$e^{ikx-i\sigma t}$	$\sigma = c_0 k$
b) $\phi_{tt} - c_0^2\phi_{xx} = 0$	$e^{ikx-i\sigma t}$	$\sigma^2 = c_0^2 k^2$
c) $\phi_t + \vec{c}_0 \cdot \nabla\phi = 0$	$e^{i\vec{k}\cdot\vec{x}-i\sigma t}$	$\sigma = \vec{c}_0 \cdot \vec{k}$
d) $\phi_{tt} - c_0^2\nabla^2\phi = 0$	$e^{i\vec{k}\cdot\vec{x}-i\sigma t}$	$\sigma^2 = c_0^2\lvert\vec{k}\rvert^2$
e) $\nabla^2\phi_t + \beta\phi_x = 0$	$e^{i\vec{k}\cdot\vec{x}-i\sigma t}$	$\sigma = -\beta k/\lvert\vec{k}\rvert^2$

Each linearized equation is a statement of approximate dynamical and thermodynamical conservation laws. All are solved using plane waves of the type discussed above. All require different dispersion relations, and the solutions have different properties. For example, for cases (a)-(d), the phase speed $c = \sigma/|\vec{k}|$ is independent of wavelength, frequency or direction. Such waves are *nondispersive* or *dispersionless* because all waves (for each case individually) travel with the same speed. In case (e), the phase speed c is dependent upon the wavelength and the direction, so these waves are dispersive. As we will see, this basically means that a group of such waves will not remain together while propagating through the medium, but instead will break up or disperse. Standing waves, as defined above, are possible in cases (b) and (d) because oppositely travelling waves can occur with the same wavenumber but with frequencies of opposite sign. That is, the dispersion relation has more than one branch, $\sigma = \Omega_j(\vec{k})$ for $j = 1, ...n$. However, in cases (a), (c) and (e), a given wavenumber corresponds to only a single frequency (only one branch), i.e. waves can travel only in one direction, so standing waves are not possible.

Several cautionary notes are in order here. Plane waves are rarely the complete solution to any boundary or initial value problem. If the medium is actually homogeneous and steady, then plane waves may often be superposed to solve such problems. However, often the medium is not homogeneous or steady, so plane wave solutions then require modifications before they can be used. We shall spend a good part of this course deriving linearized equations which isolate particular physics and we shall discuss the appropriate plane wave solutions in detail. But it must be kept in mind that, in order to establish a basis for comparison with observations of real systems, a boundary or initial value problem must be solved, most probably including medium inhomogeneities. We shall, in some instances, show examples of such problems for some sets of linearized equations.

1.3 Linear superposition of plane waves

In a homogeneous medium, initial value problems are solvable as Fourier integrals which amounts to summing an infinite number of plane wave solutions. If the dispersion relation has n branches

$$\sigma = \Omega_j(\vec{k}) \qquad j = 1, ...n$$

then n initial conditions are normally required. The solution takes the form

$$\phi(\vec{x},t) = \sum_{j=1}^{n} \int\int\int_{-\infty}^{\infty} A_j(\vec{k}) e^{i[\vec{k}\cdot\vec{x} - \Omega_j(\vec{k})t]}\, d\vec{k}$$

where the $A_j(\vec{k})$ are fixed by the initial conditions. For example, if $n = 1$, and we are in one dimension

$$\sigma = \Omega(k)$$

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i[kx - \Omega(k)t]}\, dk$$

$A(k)$ is fixed by specifying $\phi(x,0)$, that is

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k) e^{ikx}\, dk \quad ; \quad A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(x,0) e^{-ikx}\, dx$$

Notice that if $\Omega = ck$, then

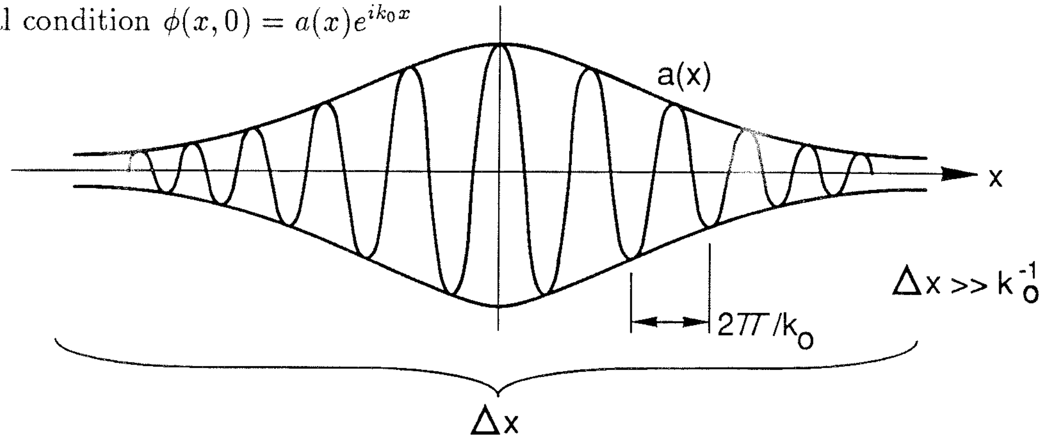
$$\phi(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx-ckt)}\, dk = \int_{-\infty}^{\infty} A(k) e^{ik(x-ct)}\, dk = \phi(x-ct,0)$$

This means that, in this special case, the initial condition $\phi(x,0)$ translates towards $x > 0$ at speed c without changing shape.

For homogeneous media, therefore, the problem is generally solved by (i) finding the dispersion relation, (ii) deducing the $A_j(\vec{k})$ from initial conditions, and (iii) evaluating a set of Fourier integrals.

1.4 The method of stationary phase: Group velocity

The greatest difficulty with the above procedure is most often that the integrals are hard to do. A very useful approximate technique with physical content is the *method* *of stationary phase.* As a preview, let us consider a one-dimensional example with the special initial condition $\phi(x,0) = a(x)e^{ik_0x}$



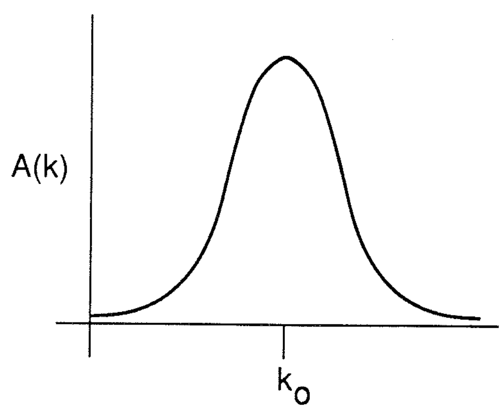
This represents a slowly modulated plane wave with envelope $a(x)$. We can always write

$$\phi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}\,dk \quad ; \quad A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x,0)e^{-ikx}\,dx$$

and so

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(x)e^{i(k_0-k)x}\,dx \quad ; \quad a(x) = \int_{-\infty}^{\infty} A(k)e^{i(k-k_0)x}\,dk$$

In the integral for $A(k)$, the contribution to the integral itself is mostly from the regions where the quantity $(k_0 - k)x$ is small. In fact, where this quantity is large, $e^{i(k_0-k)x}$ oscillates rapidly and the integrated parts cancel each other. Moreover, $a(x) = 0$ for $x \gg \Delta x$. So, $A(k)$ is centered around k_0 and peaked there for this special choice of $\phi(x,0)$.



The modulated plane wave is said to be a 'narrow band signal'.

We can evaluate $\phi(x,t)$ by expanding $\Omega(k)$ in a Taylor series about k_0:

$$\begin{aligned}
\phi(x,t) &= \int_{-\infty}^{\infty} A(k)e^{i[kx-\Omega(k)t]}\,dk \\
&\simeq \int_{-\infty}^{\infty} A(k)e^{i[kx-\Omega(k_0)t-(k-k_0)\frac{\partial\Omega}{\partial k}|_{k=k_0}t]}\,dk \\
&= \int_{-\infty}^{\infty} A(k)e^{i[kx-\Omega(k_0)t-(k-k_0)\frac{\partial\Omega}{\partial k}|_{k=k_0}t]}e^{ik_0x-ik_0x}\,dk \\
&= e^{i[k_0x-\Omega(k_0)t]}\int_{-\infty}^{\infty} A(k)e^{i(k-k_0)[x-\frac{\partial\Omega}{\partial k}|_{k=k_0}t]}\,dk
\end{aligned}$$

That is

$$\phi(x,t) = e^{i[k_0x-\Omega(k_0)t]}a(x-\frac{\partial\Omega}{\partial k}|_{k=k_0}t)$$

The modulating envelope moves at a velocity $\partial\Omega/\partial k|_{k=k_0}$, defined by the dispersion relation $\sigma = \Omega(k)$. This velocity is called the *group velocity*

$$c_g = \frac{\partial\Omega}{\partial k}|_{k=k_0}$$

and is *not*, in general, equal to the phase speed $c = \sigma/k$ of the modulated plane wave. Therefore, the dominant wavelength $\lambda = 2\pi/k_0$ has two speeds associated with it. They are the phase speed $c = \sigma/k_0 = \Omega(k_0)/k_0$ and the group velocity $c_g = \partial\sigma/\partial k|_{k=k_0} = \partial\Omega/\partial k|_{k=k_0}$. The modulated envelope thus moves *through* the phases of the underlying plane wave rather than with them.

The restriction to narrow band processes is illustrative but not necessary. Consider

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k)e^{i[kx-\Omega(k)t]}\,dk$$

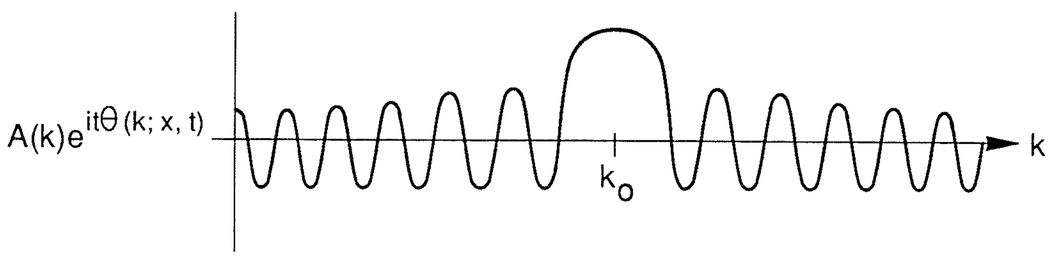
Define

$$\Theta(k;x,t) \equiv kx/t - \Omega(k)$$

Then

$$\phi(x,t) = \int_{-\infty}^{\infty} A(k)e^{it\Theta(k;x,t)}\,dk$$

The Riemann-Lebesgue theorem (e.g. Bender and Orszag, 1978, pp. 277-278) says that if $\int_{-\infty}^{\infty} A(k)\, dk$ exists, then $\lim_{t\to\infty} \int_{-\infty}^{\infty} A(k)e^{ikt}\, dk = 0$. Hence, we get little contribution to $\phi(x,t)$ unless $\Theta(k;x,t)$ has no variation with k, i.e., unless there exist k_0 such that $(\partial\Theta/\partial k)|_{k_0} = 0$. Perhaps a more intuitive statement is that the integrand looks like



in which the rapid oscillations of $e^{i\Theta t}$, as $t \to \infty$, cancel unless $\partial\Theta/\partial k = 0$ somewhere.

Stationary phase now asserts

$$\phi(x,t) \simeq \int_{-\infty}^{\infty} A(k)e^{it[\Theta(k_0)+(k-k_0)\Theta'(k_0)+(k-k_0)^2\Theta''(k_0)/2]}\, dk$$

In other words, at a given x and t, the greatest contribution to $\phi(x,t)$ is from that wavenumber k_0 at which $\Theta'(k_0;x,t) = 0$. Since $\Theta(k;x,t) = kx/t - \Omega(k)$ we have

$$x/t - \frac{\partial\Omega}{\partial k}|_{k_0} = 0$$

which means that the wavenumber k_0 that makes the biggest contribution to $\phi(x,t)$ is the one for which

$$\frac{\partial\Omega}{\partial k}|_{k_0} = x/t \quad ;$$

i.e., the one whose group velocity is x/t.

To estimate that contribution, realise that $\Theta'(k_0) = 0$, so that

$$\phi(x,t) \simeq A(k_0)e^{it\Theta(k_0)} \int_{-\infty}^{\infty} e^{i(k-k_0)^2\Theta''(k_0)t/2}\, dk$$

or, since $\int_{-\infty}^{\infty} e^{-\alpha z^2}\, dz = (\pi/\alpha)^{1/2}$, then

$$\phi(x,t) \simeq A(k_0) e^{it\Theta(k_0)} [2\pi / - it\Theta''(k_0; x, t)]^{1/2}$$

$$\phi(x,t) \simeq A(k_0) e^{i[k_0 x - \Omega(k_0)t]} [2\pi / - it\Theta''(k_0; x, t)]^{1/2}$$

The solution is thus a slowly modulated plane wave whose wavenumber k_0 is characterized by $\partial\Omega/\partial k|_{k_0} = x/t$.

The solution is only valid for very large t and x because it requires the rapid oscillation of $e^{i[kx-\Omega(k)t]}$ at all k except those where $x - \frac{\partial\Omega}{\partial k}t = 0$. It thus describes the waves far from and long after their initial generation.

1.5 Waves in slowly varying media: Ray theory

The procedure of Fourier synthesis followed by stationary phase interpretation is natural in homogeneous media. It introduces the concept of group velocity, but the idea and significance of group velocity extend into problems for which Fourier synthesis is clumsy at best. An important set of such problems includes those for which the medium varies over a scale L_m which is much greater than the length scale of the waves, L_w. In these cases, an approximate technique called the WKB method can exploit the smallness of L_w/L_m. The WKB method, however, is often tedious and difficult to interpret. Instead, a general 'recipe' called *ray theory*, which corresponds to the first and second orders of approximation of the WKB method, can be used.

Let us consider a locally periodic solution of the form

$$\phi = a(x,t) e^{i\Theta(x,t)}$$

in which the amplitude a and the phase Θ are slowly varying functions of x and t; i.e., they vary with the large space and time scales of the medium or of the wave groups and not the small scale of the sinusoidal plane wave. We can define the local wavenumber $\vec{k}$ and the local frequency N by

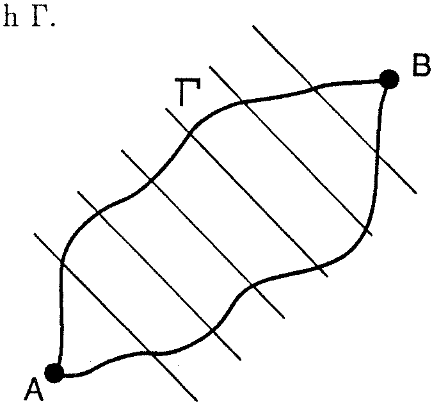
$$\vec{k} = \nabla\Theta|_t \quad ; \quad N = -\Theta_t|_x$$

where ∇ is the gradient operator and $|_t$, $|_x$ indicate that the partial derivatives are carried out keeping the other coordinate constant. Thus, $\Delta a/a \ll 1$ and $\Delta\Theta/\Theta \ll 1$ over $\vec{k}^{-1}$ and N^{-1}.

For these definitions, we see first that

$$\nabla \times \vec{k} = 0$$

which states that the local wavenumber is irrotational. Now suppose we go from place A to place B over the path Γ.



The number of wave crests we pass through is

$$n = \frac{1}{2\pi}\int_A^B \vec{k}\cdot d\vec{s}$$

But since $\oint \vec{k}\cdot d\vec{s} = \int \hat{k}\cdot\nabla\times\vec{k}\; dr = 0$ (by Stokes' theorem where $\hat{k}$ is the unit vector normal to the surface and dr is an element of the area inside the path), then the number of wave crests inside the region is conserved. That is, the crests have no ends, so the number of crests within a wave group will be the same for all time. This need not be true for all waves, but it is true for slowly varying plane waves as defined above.

From the definition of $\vec{k}$ and N, it follows that

$$\frac{\partial \vec{k}}{\partial t}|_x + \nabla N|_t = 0 \tag{1.1}$$

Now with the above definition of n, we have

$$\frac{\partial n}{\partial t} = \frac{1}{2\pi}\int_A^B \frac{\partial \vec{k}}{\partial t} \cdot d\vec{s} = -\frac{1}{2\pi}\int_A^B \nabla N \cdot d\vec{s} = \frac{1}{2\pi}(N_A - N_B)$$

This says that the rate of change of the number of wave crests between A and B is equal to the rate of crest inflow at A minus the rate of crest outflow at B. Thus (1.1) expresses the conservation of wave crests between A and B, i.e., crests are neither created nor destroyed.

So far, we have defined the local wavenumber and frequency only as derivatives of Θ. There has been no direct statement of dynamics. We introduce dynamics by *asserting* that the wavenumber and frequency must be related in just the same way that they are for a plane wave!

$$N = \Omega(\vec{k}; \vec{x}, t)$$

where, if we solved for plane waves $e^{i(\vec{k}\cdot\vec{x}-\sigma t)}$ while keeping all variable medium parameters momentarily constant, we would obtain $\sigma = \Omega(\vec{k}; \vec{x}, t)$ as our dispersion relation. This turns out to be equivalent to the lowest order of a WKB calculation, despite being stated here as an arbitrary recipe.

Now this assertion and the definitions of $\vec{k}$ and N allow us to introduce the group velocity in another way.

$$\frac{\partial N}{\partial t}|_{\vec{x}} = \frac{\partial \Omega}{\partial t}|_{\vec{k},\vec{x}} + \frac{\partial \Omega}{\partial k_i}|_{\vec{x},t}\frac{\partial k_i}{\partial t}|_{\vec{x}} = \frac{\partial \Omega}{\partial t}|_{\vec{k},\vec{x}} - c_{g_i}\frac{\partial N}{\partial x_i}|_t$$

in which the group velocity has been defined as

$$c_{g_i} \equiv \frac{\partial N}{\partial k_i} = \frac{\partial \Omega}{\partial k_i}$$

and the repeated index implies summation. In vector form, we have

$$\frac{\partial N}{\partial t} + \vec{c}_g \cdot \nabla N = \frac{\partial \Omega}{\partial t}|_{\vec{k},\vec{x}} \tag{1.2}$$

In a similar manner starting with (1.1)

$$\frac{\partial k_i}{\partial t}|_{\vec{x}} + \frac{\partial \Omega}{\partial x_i}|_{\vec{k},t} + \frac{\partial \Omega}{\partial k_j}|_{\vec{x},t}\frac{\partial k_j}{\partial x_i} = 0$$

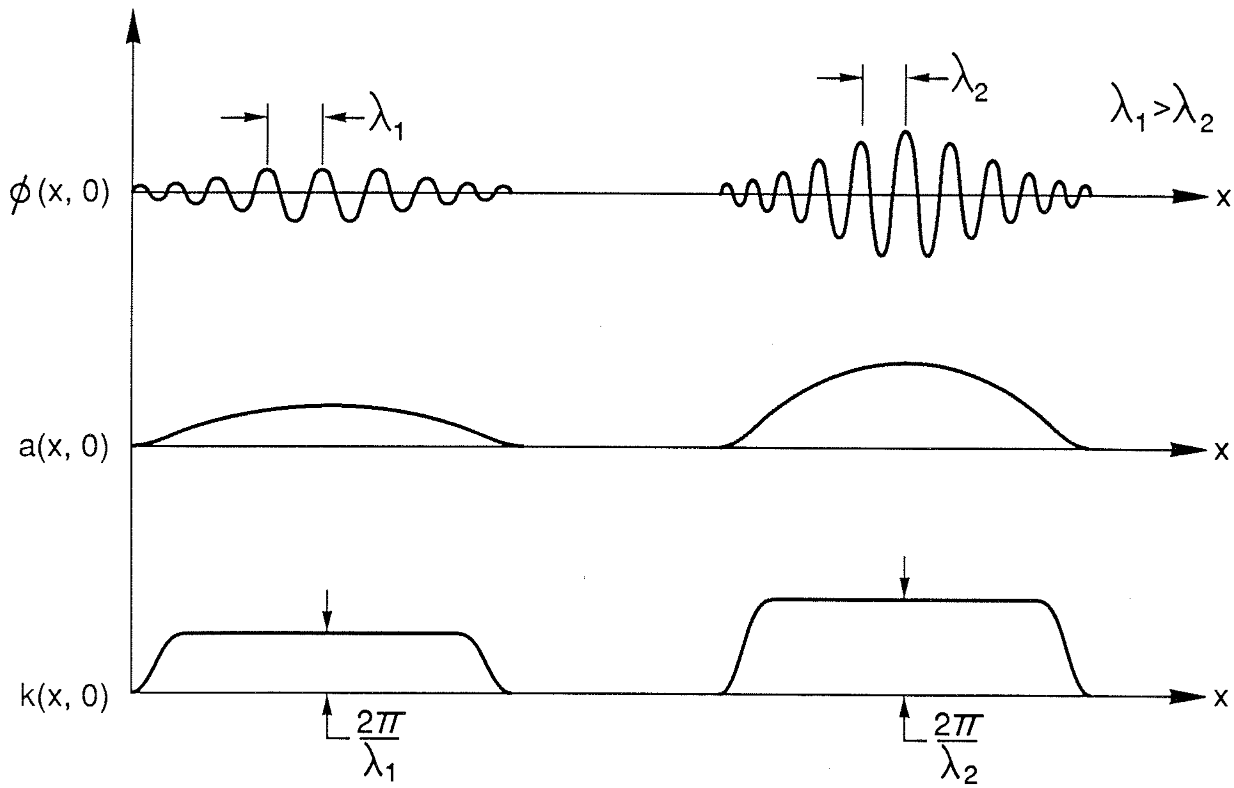
Since $\nabla \times \vec{k} = 0$, then $\partial k_j/\partial x_i = \partial k_i/\partial x_j$, so we have

$$\frac{\partial k_i}{\partial t} + \vec{c}_g \cdot \nabla k_i = -\frac{\partial \Omega}{\partial x_i}|_{\vec{k},t} \tag{1.3}$$

We thus have very simple expressions, (1.2) and (1.3), for the evolution of local wavenumber $\vec{k}$ and local frequency N as we move along a ray (i.e., we move at the local group velocity $\vec{c}_g$) in terms of the plane wave dispersion relation. Such variations occur when $\Omega(\vec{k};\vec{x},t)$ has parametric x,t dependence such as if waves move in water of variable depth.

The implications of these equations deserve some discussion. Suppose first that the medium is homogeneous, i.e. $N = \Omega(\vec{k}) \neq \Omega(\vec{k};\vec{x},t)$. One possible solution is the plane wave $\phi = ae^{i(\vec{k}\cdot\vec{x}-Nt)}$ when $\vec{k}$ and N are constants. The initial condition is $\phi(\vec{x}) = ae^{i\vec{k}\cdot\vec{x}}$. Since $\partial\vec{k}/\partial x_i \equiv 0$; $\partial\Omega/\partial x_i \equiv 0$ then from (1.3), $\partial\vec{k}/\partial t \equiv 0$ everywhere, that is $\vec{k}$ never changes at future times. Similarly, $N = \Omega(\vec{k})$ gives N at $t = 0$. Since $\partial N/\partial x_i = 0$, $\partial\Omega/\partial t = 0$, then by (1.2) $\partial N/\partial t = 0$ everywhere, that is N never changes at future times. The plane wave in a homogeneous medium is thus entirely consistent with the ray theory formulation.

Suppose now that the medium remains homogeneous, but the initial conditions are more complicated. Both a and $\vec{k}$ have slow x dependence at $t = 0$ as illustrated below:



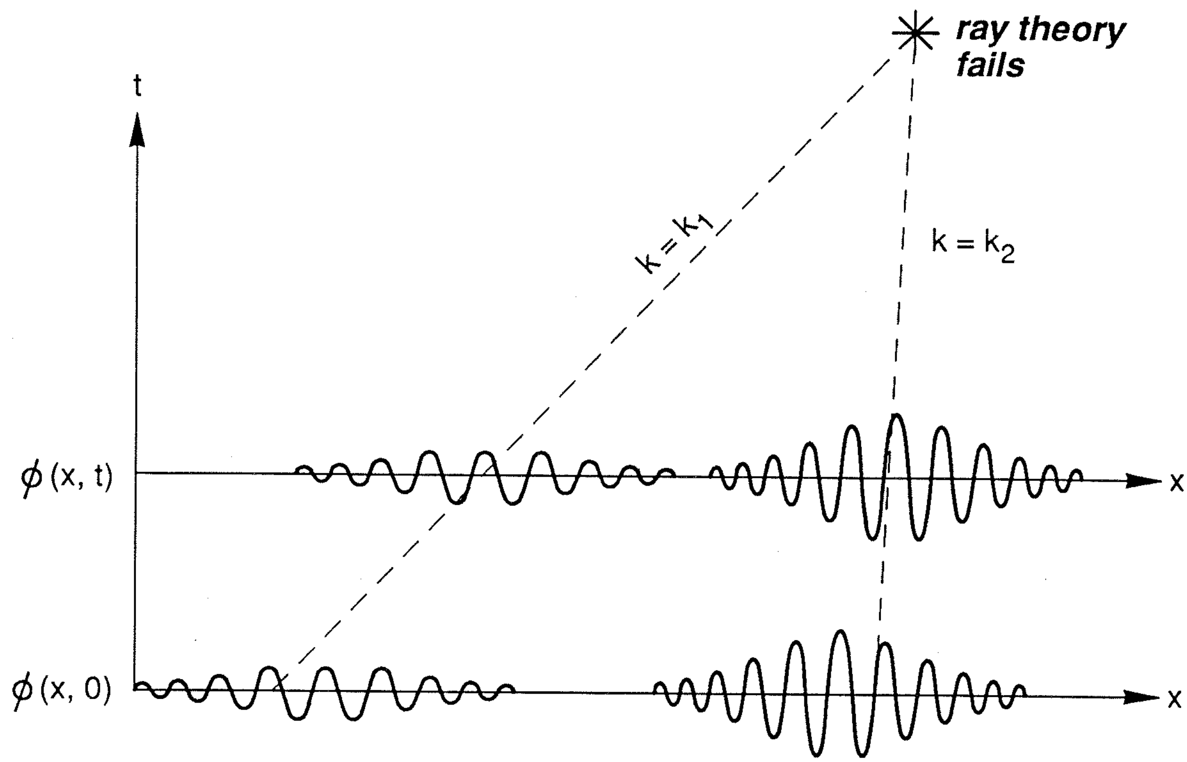
Notice that a and $\vec{k}$ should vary slowly over λ, even though the sketch is not very slowly varying.

The initial frequency is obtained from $N(x,0) = \Omega[\vec{k}(\vec{x},0)]$. To find $N(\vec{x},t)$, $\vec{k}(\vec{x},t)$ we solve the initial value problem

$$\frac{\partial k_i}{\partial t} + c_{g_j}\frac{\partial k_i}{\partial x_j} = 0$$

$$\frac{\partial N}{\partial t} + c_{g_j}\frac{\partial N}{\partial x_j} = 0$$

because the assumed homogeneity of the medium implies $\partial\Omega/\partial t = 0$ and $\partial\Omega/\partial x_i = 0$. This initial value problem may have to be solved numerically, but the equations have a simple physical interpretation. They say that, if we move at the group velocity $\vec{c}_g = \nabla_{\vec{k}}\Omega$ appropriate to the wavenumber $\vec{k}$ and the frequency $N = \Omega(\vec{k})$, then we shall see no change in N and $\vec{k}$ at future times. In other words, N and $\vec{k}$ are constant following a group in a homogeneous medium. The situation can be sketched as follows



Clearly, if we sit at a fixed $\vec{x}$, different groups pass at different times. So at fixed $\vec{x}$, $\partial N/\partial t \neq 0$, $\partial \vec{k}/\partial t \neq 0$, in general, even though the medium is homogeneous. The whole idea fails if the rays, given by

$$\vec{x} = \vec{x}_0 + \int_0^t \vec{c}_g[\vec{k}(\vec{x}, t)]\, dt$$

cross each other. In that case, the solution is no longer of slowly varying form.

From this point of view, the medium inhomogeneities are only technical complications. In the general inhomogeneous case, we must solve (1.2) and (1.3), so $\vec{k}$ and N vary even though we move with a group. If we define a 'total' derivative as

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{c}_g \cdot \nabla$$

which is the derivative following the wave group (or wave packet), then (1.2) and (1.3) can be rewritten as

$$\begin{aligned}\frac{dk_i}{dt} &= -\frac{\partial \Omega}{\partial x_i}\\ \frac{dN}{dt} &= \frac{\partial \Omega}{\partial t}\end{aligned}$$

while the position of the wave group is given by

$$\frac{d\vec{x}}{dt} = \vec{c}_g[\vec{k}(\vec{x}, t)]$$

Then we have a set of three ordinary differential equations for $\vec{x}$ (position of the wave packet), $\vec{k}$ and N. These may be integrated in time from a number of different starting positions $\vec{x}_0$ in order to get $\vec{k}$, N at future times, a procedure which is computationally efficient and effective. The path $d\vec{x}/dt = \vec{c}_g$ defines the ray.

The lowest order of the corresponding WKB calculation justifies the foregoing assertions. The next order of the WKB calculation fixes the amplitude. In many cases, the more complex WKB calculation amounts to solving

$$\frac{\partial A}{\partial t} + \nabla \cdot (\vec{c}_g A) = 0$$

where $A \equiv \varepsilon/N$ and ε is the wave energy. A is called the *action* of the wave. Usually $\varepsilon \propto a^2$ so this equation really describes a, but a great deal of further discussion is necessary to establish its validity. Here we have simply set forward 'recipes' which give a, N, $\vec{k}$.

Chapter 2

Acoustic waves

Being now equipped with some ideas about wave motions, it is useful to consider an example of waves which occurs in both the ocean and the atmosphere and which can illustrate many of the ideas in a rather simple way. Acoustic or sound waves, as Lighthill (1978) points out, are the most fundamental waves in fluids because they can exist in the absence of any external force field. Instead of gravity or rotation, for example, providing a restoring force for the motions, the restoring force for acoustic waves is the fluid's resistance to compression (i.e., its compressibility).

2.1 Basic physics

When viscous dissipation, rotation and gravitational forces are neglected, the momentum and continuity equations are

$$\frac{\partial \vec{u}^*}{\partial t} + \vec{u}^* \cdot \nabla \vec{u}^* = -\frac{1}{\rho^*}\nabla p^*$$

$$\frac{\partial \rho^*}{\partial t} + \nabla \cdot (\rho^* \vec{u}^*) = 0$$

An equation relating density and pressure may be obtained from the first law of thermodynamics (Batchelor, 1967; Chapter 3). It can be shown that, if $\rho^* = \rho^*(p^*, T)$ and the motions are adiabatic so that $\partial S/\partial t = 0$ where S is the entropy, then $\rho^* = \rho^*(p^*, S)$ and

$$\left(\frac{D\rho^*}{Dt}\right)_S = \left(\frac{\partial\rho^*}{\partial p^*}\right)_S \left(\frac{Dp^*}{Dt}\right)_S$$

A solution of these equations, although trivial, is

$$\rho^* = \rho_0 \; ; \; p^* = p_0 \; ; \; \vec{u}^* = 0$$

This solution is not very exciting, so we would like to study small deviations from it. Thus, we write

$$\rho^* = \rho_0 + \rho \; ; \; p^* = p_0 + p \; ; \; \vec{u}^* = 0 + \vec{u}$$

where $\rho, p, \vec{u}$ are of infinitesmal amplitude. After substituting into the original equations and neglecting products of small quantities, we have

$$\frac{\partial\vec{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p$$

$$\frac{\partial\rho}{\partial t} + \rho_0\nabla\cdot\vec{u} = 0$$

$$\frac{\partial\rho}{\partial t} = c^{-2}\frac{\partial p}{\partial t}$$

where $c^2 = (\partial\rho^*/\partial p^*)_S^{-1}$. After eliminating $\vec{u}$ and ρ in favor of p, we obtain

$$\frac{\partial^2 p}{\partial t^2} - c^2\nabla^2 p = 0$$

We recognize this as a wave equation which was listed in Chapter 1. It is easy to show that the other variables ρ, u, v, w each satisfy a similar equation.

2.2 Plane waves

Consider a homogeneous medium; $c(\vec{x}, t) = c_0$. Then $p = e^{-i\sigma t + ikx + i\ell y + imz}$ solves the wave equation provided

$$\sigma^2 = c_0^2(k^2 + \ell^2 + m^2)$$

which is the dispersion relation. For fixed σ, the locus of allowed wavenumbers in k, ℓ, m space is a sphere of radius σ/c_0. All wavenumbers $\vec{k} = k\hat{i} + \ell\hat{j} + m\hat{k}$ extending from the center of this sphere to its surface are allowed. In a given plane wave, phases propagate along the wavenumber vector at speed c_0; that is, $\sigma/|\vec{k}| = c_0$, so the waves are nondispersive. The group velocity $\vec{c}_g$ is defined by

$$c_{g_x} = \frac{\partial \sigma}{\partial k} \; ; \; c_{g_y} = \frac{\partial \sigma}{\partial \ell} \; ; \; c_{g_z} = \frac{\partial \sigma}{\partial m}$$

and it is easy to show that $|\vec{c}_g| = c_0$.

If $p = ae^{i(\vec{k}\cdot\vec{x} - \sigma t)}$, then the momentum equations say $-i\sigma\vec{u} = -i\vec{k}p/\rho_0$, or

$$\vec{u} = \frac{\vec{k}a}{\sigma\rho_0} e^{i(\vec{k}\cdot\vec{x} - \sigma t)}$$

This means that $\vec{u}$ and $\vec{k}$ are parallel, i.e. these are longitudinal waves (displacement is parallel to the direction of wave propagation). Also, $\vec{u}$ and p are in phase in this travelling plane wave.

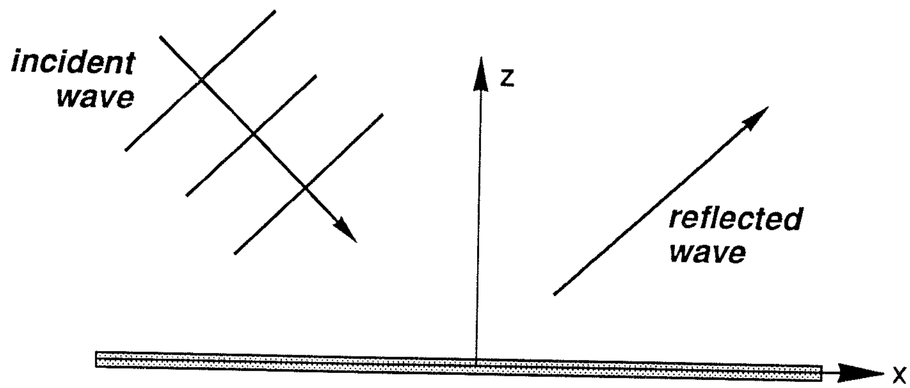
2.3 Reflection at a solid boundary

Suppose a plane wave of the form

$$p_{inc} = p_0 e^{-i\sigma t + ikx + i\ell y - imz}$$

is incident upon a solid boundary. At the solid boundary, the normal velocity must vanish; $\vec{u} \cdot \hat{n} = 0$ which means that $\nabla p \cdot \hat{n} = 0$. If the solid boundary is at $z = 0$, then the boundary condition is

$$p_z = 0 \qquad \text{at} \quad z = 0$$



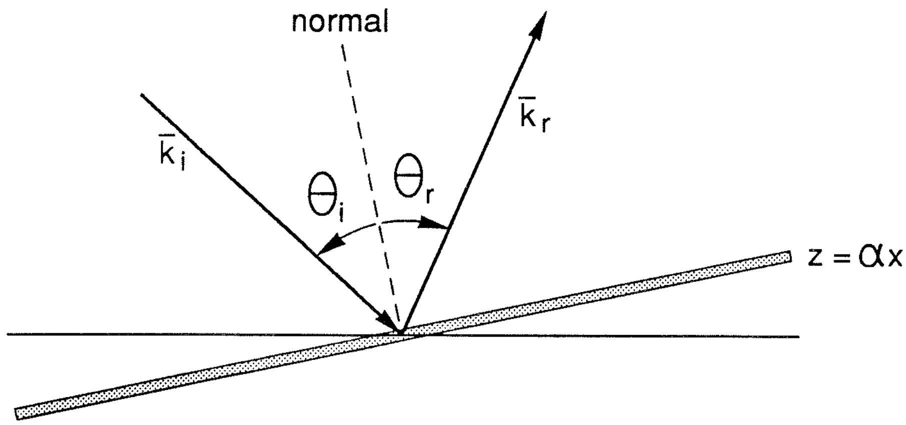
To satisfy this boundary condition, we must add a reflected wave

$$p_{ref} = p_0 e^{-i\sigma t + ikx + i\ell y + imz}$$

to the incident wave. The solution is

$$p = p_{inc} + p_{ref} = 2p_0 e^{-i\sigma t + ikx + i\ell y} \cos mz$$

Suppose the solid boundary is tilted, say $z = \alpha x$, and the incident energy approaches along $\vec{k}_i$ while the reflected energy travels along $\vec{k}_r$.



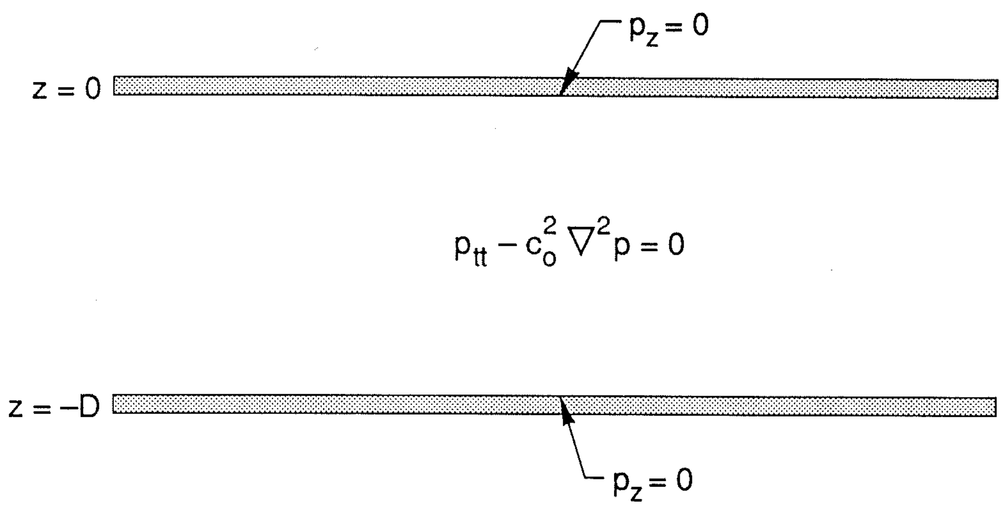
If p_{inc} and p_{ref} are to sum such that $p = p_{inc} + p_{ref}$ satisfies $\partial p/\partial n = 0$ at the solid boundary, then we must have

$$|\vec{k}_i| \cos\theta_i = |\vec{k}_r| \cos\theta_r$$

i.e., the projection of the incident wavenumber on the boundary must equal the projection of the reflected wavenumber on the boundary. But we know that $|\vec{k}_i| = |\vec{k}_r| = \sigma/c_0$, so $\theta_r = \theta_i$. That is, the reflection of these waves is specular. (This is not true of all waves, however.) Note that $\partial p/\partial n = 0$ at the boundary means $\vec{u} \cdot \hat{n} = 0$ there, so $\vec{u}_{inc} \cdot \hat{n} = -\vec{u}_{ref} \cdot \hat{n}$.

2.4 Plane waves in a channel

A very important aspect of wave motion is the effect of boundaries which form a channel or *waveguide.* Thus far, the plane waves we have considered have not been restricted in the choice of wavenumbers. That is, the entire continuum of k, ℓ, m choices has been available, provided we were willing to accept whatever frequency was required by the dispersion relation. We saw that the form of the plane wave was altered somewhat due to the presence of one boundary, so now we consider the effect of a second boundary.



Now the field equation is still valid in the interior of the channel, but the free waves must satisfy $\partial p/\partial z = 0$ on both boundaries, at $z = 0, -D$. To find a solution, we assume that the waves are free to travel along the channel, but that the cross-channel dependence is unknown.

$$p(x, y, z, t) = p_0 e^{-i\sigma t + ikx + i\ell y} P(z) \quad .$$

This is substituted into the field equation to obtain an equation for the cross-channel structure

$$P_{zz} + (\sigma^2/c_0^2 - k^2 - \ell^2)P = 0$$

$$P_z = 0 \quad \text{at} \quad z = 0, -D$$

This equation has the solution

$$P(z) = \cos n\pi z/D$$

provided that

$$\sigma^2/c_0^2 = k^2 + \ell^2 + n^2\pi^2/D^2 \qquad n = 0, 1, 2...$$

Notice that these solutions are each a sum of two plane waves

$$1/2p_0e^{-i\sigma t+ikx+i\ell y+in\pi z/D} + 1/2p_0e^{-i\sigma t+ikx+i\ell y-in\pi z/D}$$

which satisfy $\partial p/\partial z = 0$ at $z = 0$ regardless of whether n is an integer or not. However, to satisfy $\partial p/\partial z = 0$ at $z = -D$, we need $n = 0, 1, 2....$ These solutions are called *waveguide modes.*

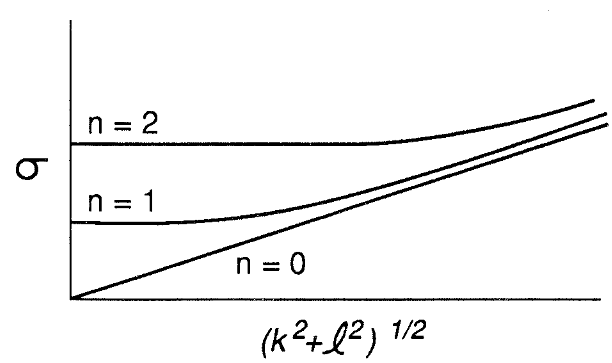
Another important point to notice is that each three-dimensional plane wave by itself satisfies the dispersion relation, so that both waves are the usual nondispersive plane waves if we think of n as continuously variable. Yet the solution viewed as a two-dimensional plane wave restricted to the channel direction is dispersive!

$$c_{ph} = \sigma/(k^2 + \ell^2)^{1/2} = \pm c_0[1 + n^2\pi^2/(k^2 + \ell^2)D^2]^{1/2}$$

The horizontal group velocity $\partial\sigma/\partial k$, $\partial\sigma/\partial\ell$ is

$$\vec{c}_g = c_0(k\hat{i} + \ell\hat{j})/(k^2 + \ell^2 + n^2\pi^2/D^2)^{1/2}$$

It is parallel to the horizontal wavenumber $k\hat{i} + \ell\hat{j}$ but not equal to the phase velocity.



The $n = 0$ mode actually is nondispersive.

If we fix the horizontal wavelength $2\pi/(k^2 + \ell^2)^{1/2}$, say by a wavemaker of fixed size perhaps but variable frequency, then there is an infinity of waveguide modes

$n = 0, 1, 2...$ of ever increasing frequency $\sigma^2 = c_0^2(k^2 + \ell^2 + n^2\pi^2/D^2)$. However, if we fix the frequency, then

$$(k^2 + \ell^2) = \sigma^2/c_0^2 - n^2\pi^2/D^2$$

and only for $n = 0, 1..n_{max}$ will $k^2 + \ell^2 > 0$ where $n_{max} = \text{int}[(D/\pi)(\sigma/c_0)]$. That is, only for $n = 0, 1..n_{max}$ will the waveguide modes propagate down the channel! For example, consider waves in the x-direction only

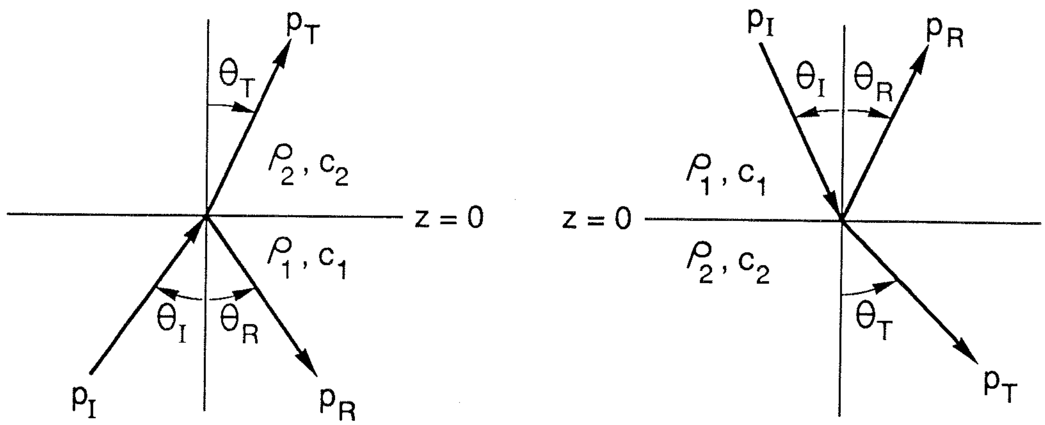
$$n < n_{max} \qquad p = e^{-i\sigma t + i(\sigma^2/c_0^2 - n^2\pi^2/D^2)^{1/2}x} \cos\frac{n\pi z}{D}$$

$$n > n_{max} \qquad p = e^{-i\sigma t - (n^2\pi^2/D^2 - \sigma^2/c_0^2)^{1/2}x} \cos\frac{n\pi z}{D}$$

The first set represents travelling waves. The second set represents *evanescent* waves which decay exponentially away from their source. Practically, this means that if we have a harmonic wavemaker in the channel, then we may expect to see more cross-channel structure near the wavemaker than far away from it.

2.5 Scattering at a discontinuity

We have considered the effect of a solid boundary on the propagation of sound waves. Suppose, however, that a plane wave encounters a boundary between two fluids at which the properties change abruptly, i.e. a discontinuity.



This discontinuity could represent the air-sea interface or the ocean bottom (which is not truly a solid boundary because it transmits sound waves). In both cases the incident wave approaches the discontinuity while travelling through the medium which has density ρ_1 and phase speed c_1. The density of the medium on the other side is ρ_2 while the phase speed is c_2.

For the case on the left (upward propagating incident wave), the incident, reflected and transmitted waves have the following forms;

$$
\begin{aligned}
p_I &= ae^{-i\sigma t+ikx+im_1 z} \\
p_R &= Rae^{-i\sigma t+ikx-im_1 z} \\
p_T &= Tae^{-i\sigma t+ikx+im_2 z}
\end{aligned}
$$

where R is the reflection coefficient and T is the transmission coefficient. Notice that the incident and reflected waves have the same wavenumber component in z but that they propagate in opposite directions. The transmitted wave has a different wavenumber in z because the medium has different properties. The wavenumber in the direction of the boundary x as well as the frequency σ are the same for all three waves because there is nothing in the fluids which would change them.

To solve the problem, we require that the pressure as well as the velocity normal to the boundary w be continuous across the boundary. That is

$$p_I + p_R = p_T \quad \text{at} \quad z = 0$$

$$\frac{1}{\rho_1}(p_{Iz} + p_{Rz}) = \frac{1}{\rho_2}p_{Tz} \quad \text{at} \quad z = 0$$

Now, substituting the expressions for p_I, p_R and p_T, we obtain

$$1 + R = T$$

$$\frac{m_1}{\rho_1}(1 - R) = \frac{m_2}{\rho_2}T$$

From the dispersion relation, $m = \sigma \cos\theta / c$ which changes the second matching condition to

$$\frac{1}{\rho_1 c_1}(1 - R)\cos\theta_I = \frac{1}{\rho_2 c_2}T\cos\theta_T$$

These can be combined to yield

$$R = \frac{\rho_2 c_2 \cos\theta_I - \rho_1 c_1 \cos\theta_T}{\rho_2 c_2 \cos\theta_I + \rho_1 c_1 \cos\theta_T}$$

$$T = \frac{2\rho_2 c_2 \cos\theta_I}{\rho_2 c_2 \cos\theta_I + \rho_1 c_1 \cos\theta_T}$$

Identical expressions for R and T result for the downward propagating incident wave.

We see from these expressions that if the density times the phase speed of the second medium is much less than that of the first, $\rho_2 c_2 \ll \rho_1 c_1$, then the transmission coefficient vanishes and the reflection coefficient goes to unity, $T \to 0, R \to -1$. This is consistent with the result we obtained for a solid boundary. It is also nearly the case for the boundary between the ocean and the atmosphere where ρc is about 1.5×10^6 kg m^{-2} s^{-1} for the ocean and 400 kg m^{-2} s^{-1} for the atmosphere. So, very little sound is transmitted from the ocean to the atmosphere. On the other hand, a sound wave in the atmosphere is actually amplified upon encountering the ocean. That is, if medium

1 is the atmosphere, then $T \to 2$. Of course, the sound wave in the atmosphere travels so slowly relative to the ocean that its energy flux is generally fairly small, so the amplification is a rather small effect as well. In either case, the energy flux in the z direction is conserved because

$$|p_I w_I| = |p_R w_R| + |p_T w_T|$$

To complete the calculation, we must find the angle of the transmitted wave, θ_T. This is found by writing the frequency on both sides of the discontinuity as

$$\sigma = c_1(k^2 + m_1^2)^{1/2} = c_2(k^2 + m_2^2)^{1/2}$$

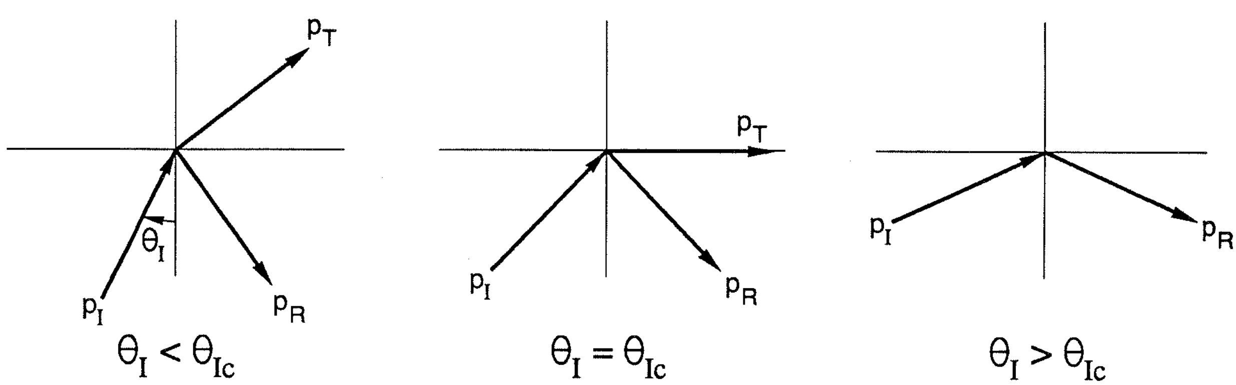
We can write this in terms of the wave angles since $(k^2 + m^2)^{1/2} = k/\sin\theta$. Thus,

$$\frac{\sin\theta_I}{c_1} = \frac{\sin\theta_T}{c_2}$$

which is known as *Snell's Law*. From this we see that, if $c_1 < c_2$, then there exists a critical angle of incidence

$$\theta_{Ic} = \sin^{-1}(c_1/c_2)$$

beyond which there is total reflection of the incident wave despite the fact that the second medium can support sound waves. The boundary is then effectively solid.



This is called total internal reflection.

2.6 Generation of plane waves

At this point, it is natural to ask how these plane waves may be generated. What initial or boundary conditions or forcing terms are needed to generate solutions of the wave equation corresponding to some physical situation? If there are wavemakers in the medium, they can be modelled by body forces $\vec{F}/\rho_0$ and mass sources Q

$$\vec{u}_t = -\nabla p/\rho_0 + \vec{F}/\rho_0$$

$$\rho_t + \rho_0 \nabla \cdot \vec{u} = Q$$

Combining these with $p_t = c^2 \rho_t$ yields

$$p_{tt} - c^2 \nabla^2 p = c^2 (Q_t - \nabla \cdot \vec{F})$$

We can now consider two types of problems: initial value problems and those forced from rest. In both types we solve the homogeneous wave equation while satisfying $\partial p/\partial n = 0$ on the solid boundaries and requiring outgoing waves at infinity, i.e. a *radiation condition*. For the initial value problems, p and p_t are specified at time $t = 0$, while for those forced from rest they are set to zero. Of course, there is not really a fundamental distinction because solutions of one type may be linearly superposed to obtain solutions to the other type. The solution procedures may, however, be quite different.

2.6.1 An initial value problem

Let us consider a one-dimensional initial value problem

$$p_{tt} - c^2 p_{xx} = 0 \qquad -\infty < x < \infty$$

$$p(x,0) = P_0(x) \;\; ; \;\; p_t(x,0) = Q_0(x)$$

We will solve this be the *method of characteristics.* The most general solution is

$$p = f(x - ct) + g(x + ct)$$

To satisfy the initial conditions

$$\begin{aligned} f(x) + g(x) &= P_0(x) \\ -cf'(x) + cg'(x) &= Q_0(x) \end{aligned}$$

The second integrates to $f(x) - g(x) = -1/c \int_0^x Q_0(x')\, dx' + K$ whence

$$\begin{aligned} 2f(x) &= P_0(x) - 1/c \int_0^x Q_0(x')\, dx' + K \\ 2g(x) &= P_0(x) + 1/c \int_0^x Q_0(x')\, dx' - K \end{aligned}$$

These give the solution as

$$p(x,t) = 1/2[P_0(x - ct) + P_0(x + ct) + 1/c \int_{x-ct}^{x+ct} Q_0(x')\, dx']$$

Note that $p(x,t)$ depends only on the initial conditions over the range $x \pm ct$.

If $Q_0(x) \equiv 0$, then the solution is very simple

$$p(x,t) = 1/2[P_0(x - ct) + P_0(x + ct)]$$

for which case the solution could have been obtained using the Fourier method, although it is not the method of choice in this problem. Set

$$p(x,t) = \int_{-\infty}^{\infty} \bar{p}(k,t) e^{ikx}\, dk$$

Then

$$\bar{p}_{tt} + c^2 k^2 \bar{p} = 0$$

$$\bar{p}(k,0) = \bar{P}_0(k) \;\; ; \;\; \bar{p}_t(k,0) = 0$$

The solution to this problem is

$$\bar{p}(k,t) = \bar{P}_0(k)\cos(ckt)$$

from which

$$\begin{aligned} p(x,t) &= \int_{-\infty}^{\infty} \bar{P}_0(k)\cos(ckt)e^{ikx}\,dk \\ &= 1/2\int_{-\infty}^{\infty} \bar{P}_0(k)(e^{ikx+ickt} + e^{ikx-ickt})\,dk \\ &= 1/2[P_0(x-ct) + P_0(x+ct)] \end{aligned}$$

The integration is trivial in this case but not always.

2.6.2 Forcing from rest

Assume that the forcing has the rather simple form

$$Q_t - \nabla\cdot\vec{F} = \delta(x)q_t(t)$$

where $q(t) = q_t(t) = 0$ for $t < 0$ and q_t is finite. Now we solve

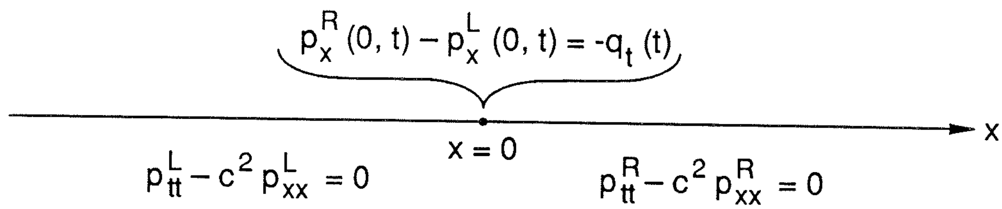
$$p_{tt} - c^2 p_{xx} = \delta(x)q_t(t)c^2$$

$$p(x,0) = p_t(x,0) = 0 \quad \text{at} \quad t = 0$$

We may put the forcing into the boundary condition by

$$\int_{0-}^{0+} (p_{tt} - c^2 p_{xx})\,dx = -c^2 p_x|_{0-}^{0+} = c^2 q_t(t)$$

That is, $p_x(x = 0+, t) - p_x(x = 0-, t) = -q_t(t)$ so that the forcing at $x = 0$ is interpretable as a specified discontinuity there. So we must solve



where p^L and p^R are solutions on the left and right of the discontinuity, respectively. Most generally, p^L and p^R are functions of $x \pm ct$. We write them along with the requirement of symmetry

$$p^R(x,t) = p^L(-x,t)$$

$$p^R(x,t) = f(x - ct) + g(x + ct)$$

$$p^L(x,t) = f(-x - ct) + g(-x + ct)$$

Imposing the jump condition at $x = 0$ yields

$$f'(-ct) + g'(ct) + f'(-ct) + g'(ct) = -q_t(t)$$

but this does not specify f and g. To specify them, we must impose a radiation condition, i.e.,

$$p^R(x,t) = f(x - ct) \qquad p^R \text{ is all right going waves}$$

$$p^L(x,t) = f(-x - ct) \qquad p^L \text{ is all left going waves}$$

Now we have

$$\begin{aligned} 2f'(-ct) &= -q_t(t) \\ 2f(-ct) &= q(t)c \\ f(\tau) &= \frac{c}{2}q(-\tau/c) \end{aligned}$$

from which

$$
\begin{aligned}
p^R(x,t) &= \frac{c}{2}q(-x/c+t) \\
p^L(x,t) &= \frac{c}{2}q(x/c+t)
\end{aligned}
$$

Thus, forcing at the origin is modelled as a jump in p_x and we must assume that all of the motion is *away* from the source in order to get a unique answer.

If the forcing were harmonic with $q(t) = e^{-i\sigma t}/(-i\sigma)$ then

$$p_{tt} - c^2 p_{xx} = c^2\delta(x)e^{-i\sigma t}$$

and the solution would be

$$
\begin{aligned}
p^R(x,t) &= \frac{-c}{2(i\sigma)}e^{-i\sigma(-x/c+t)} \\
p^L(x,t) &= \frac{-c}{2(i\sigma)}e^{-i\sigma(x/c+t)}
\end{aligned}
$$

In other words, plane waves radiating outwards from $x = 0$. The radiation condition that we imposed models a little bit of dissipation in the sense that the solution looks dissipationless locally, but nothing is reflected from $|x| \to \infty$ because even small dissipation attenuates any reflected waves over a long distance. We could, in fact, add a friction term to the momentum equations and solve again to obtain a solution which would become the present solution for vanishingly small friction.

2.7 Slowly varying medium

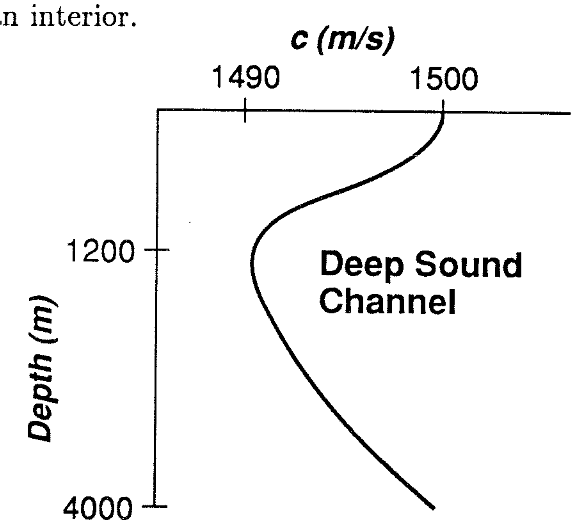
We have considered cases in which the speed of sound remains constant in the medium or changes abruptly at an interface. However, the speed of sound within the ocean varies in space because the ocean is not a uniform fluid. In fact the sound speed in the ocean is sensitive to the temperature, salinity and pressure of the ocean and may be described by the following empirical formula:

$$c(s,T,z) = c_0+\alpha_0(T-10)+\beta_0(T-10)^2+\gamma_0(T-18)^2+\delta_0(s-35)+\epsilon_0(T-18)(s-35)+\zeta_0|z|$$

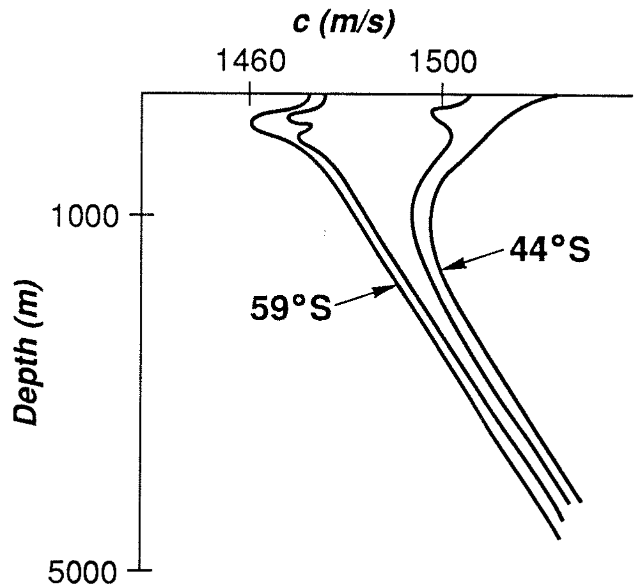
where the coefficients have the appropriate mks units and have values of

$$c_0 = 1493.0,\quad \alpha_0 = 3.0,\quad \beta_0 = -0.006,\quad \gamma_0 = -0.04,\quad \delta_0 = 1.2,\quad \epsilon_0 = -0.01,\quad \zeta_0 = 0.0164$$

This says that the speed of sound varies quadratically with temperature, and linearly with salinity and depth. The depth effect is due to changes in the ambient pressure. For typical ocean conditions, the temperature effect dominates in the shallow water, while the pressure effect dominates in the deep water. The sound speed increases with an increase in either temperature or depth, so there is typically a sound speed minimum in the ocean interior.



The situation is different in the arctic where there is little effect of warming near the surface. There the sound speed tends to decrease right up to the surface.



We can examine the effects of these variations in sound speed by applying our knowledge of ray theory. We must assume that the wavelengths of the acoustic waves are much less that the scale over which the sound speed changes. That is, the wavelength must be small compared to the total ocean depth. We will consider only two dimensions, the vertical and one horizontal. Recalling our discussion of ray theory, we write the dispersion relation as

$$\sigma = \Omega(k, m; z)$$

Since the medium varies only in z, we have $\partial\Omega/\partial x = 0$, $\partial\Omega/\partial t = 0$ but $\partial\Omega/\partial z \neq 0$. Thus, the ray equations become

$$\begin{aligned} \frac{dN}{dt} &= 0 \\ \frac{dk}{dt} &= 0 \\ \frac{dm}{dt} &= -\frac{\partial\Omega}{\partial z} \end{aligned}$$

These say that the component of the wavenumber in the x direction remains constant in time (which makes sense since the medium varies only in z), and that the frequency remains fixed at the initial frequency. We could integrate these following along a ray with the group velocity, but we will examine the qualitative behavior by considering Snell's Law which can be derived in the same manner as for the case of scattering at the discontinuity. The frequency can be written in terms of the angle that the wavenumber makes with the vertical to obtain

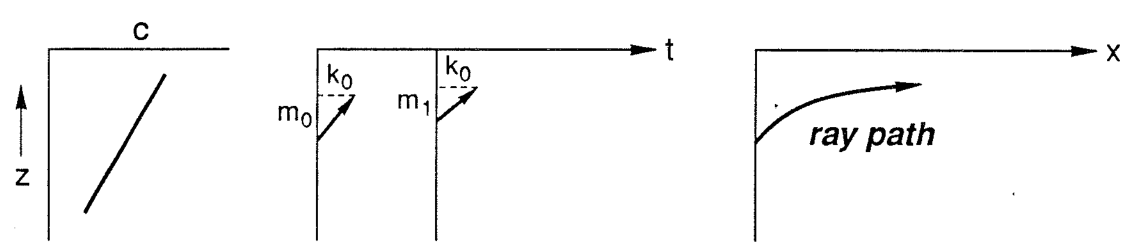
$$\frac{\sin\theta_0}{c_0} = \frac{\sin\theta}{c(z)}$$

or

$$\sin\theta = \frac{c(z)}{c_0}\sin\theta_0$$

where c_0 and θ_0 are the initial values.

Consider the case in which the sound speed decreases with depth, $c = c_0(1+\alpha z)$ (remember that z is positive upwards). This means that a wave moving upward moves into a region of increasing sound speed, so the angle with the vertical must increase as well. Thus, the wave moves toward a horizontal path. This may also be seen from the ray equations where $-\partial\Omega/\partial z < 0$, so that m must decrease with upward motion. Decreasing m leads to a more horizontal propagation path.



Similarly if c increases in the deep ocean, sound waves moving downward will be turned toward the horizontal.

When the ray becomes nearly horizontal, ray theory must be applied very carefully. From the ray definition, we can write

$$\frac{dz}{dx} = \frac{c_{gz}}{c_{gx}} = \frac{\partial\Omega/\partial m}{\partial\Omega/\partial k} = \frac{m}{k} = \left(\frac{\sigma^2}{c^2(z)k^2} - 1\right)^{1/2}$$

which gives the slope of the ray path. This may be approximated near the critical level z_c by expanding in a Taylor series to obtain

$$\frac{dz}{dx} \simeq \left[(z - z_c) \frac{d}{dz} \left(\frac{\sigma^2}{c^2(z)k^2} \right)_{z=z_c} \right]^{1/2}$$

which integrates to

$$z = z_c + \frac{1}{4} \left[\frac{d}{dz} \left(\frac{\sigma^2}{c^2(z)k^2} \right) \right]_{z=z_c} (x - x_0)^2$$

Thus, the ray path is parabolic near the critical level, so an upward propagating ray turns downward.

Similarly, a downward propagating ray which encounters an increasing c at depth will eventually turn upward (provided it does not intersect the bottom). The end result is that the minimum in the sound speed acts as a sound channel where acoustic energy can propagate over hundreds of kilometers without encountering the bottom provided the incidence angle is not too oblique. Numerous examples are reproduced in Apel (1987). This is the basis of acoustic tomography, in which this efficient propagation is used to infer properties of the ocean. Sound waves are generated at a source and received at a listening station. For a fixed vertical profile of the sound speed, the rays may be calculated using ray theory. The received signal is then compared with that expected for a horizontally uniform medium, and differences are used to deduce various physical phenomena which might have occurred along the ray paths. This is generally called an *inverse problem* because boundary observations are used to determine the interior physics, rather than the reverse.

Chapter 3

Surface gravity waves

Probably the most familiar form of wave motion with which we have extensive experience is surface gravity waves. This class of waves includes most of the waves which occur on the interface between the atmosphere and a body of water, be it the ocean, a lake or a puddle. The restoring force which makes such waves possible is gravity – hence the name.

3.1 Homogeneous medium

Let us consider an inviscid, incompressible, homogeneous fluid bounded by a free surface near $z = 0$ and a flat bottom boundary at $z = -D$.