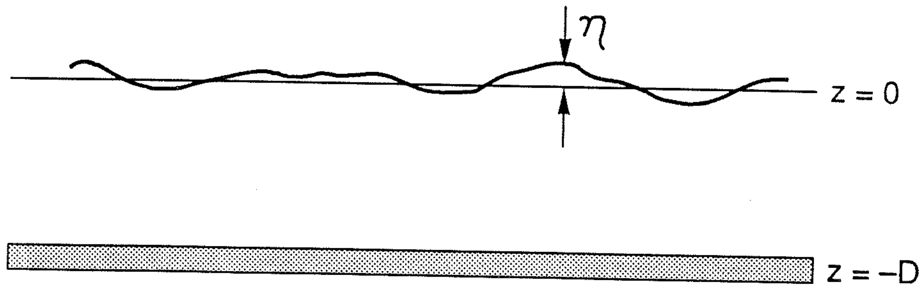
# Chapter 3

# Surface gravity waves

Probably the most familiar form of wave motion with which we have extensive experience is surface gravity waves. This class of waves includes most of the waves which occur on the interface between the atmosphere and a body of water, be it the ocean, a lake or a puddle. The restoring force which makes such waves possible is gravity – hence the name.

## 3.1 Homogeneous medium

Let us consider an inviscid, incompressible, homogeneous fluid bounded by a free surface near $z = 0$ and a flat bottom boundary at $z = -D$.




Because the fluid is inviscid

$$D\vec{u}/Dt = -\nabla p/\rho - g\hat{k}$$

If the vorticity is defined as

$$\vec{\omega} = \nabla \times \vec{u}$$

then we may take the curl ($\nabla\times$) of the momentum equations to obtain

$$D\vec{\omega}/Dt = (\vec{\omega} \cdot \nabla)\vec{u}$$

In this form, we see that if initially $\vec{\omega}(\vec{x}, 0) = 0$ everywhere, then $\vec{\omega}(\vec{x}, t) = 0$ forever. We therefore suppose that the motions we consider are generated without making $\vec{\omega}$ nonzero, so that $\nabla \times \vec{u} = 0$. This being the case, we can define a *velocity potential* by

$$\vec{u}(\vec{x}, t) = \nabla\phi(\vec{x}, t)$$

Since the fluid is incompressible, $\nabla \cdot \vec{u} = 0$, so

$$\nabla^2\phi = 0$$

The boundary conditions are derived as follows. At the bottom, $z = -D$, we require that $w = 0$, i.e.

$$\phi_z = 0 \qquad \text{at} \qquad z = -D$$

The free surface is made up of fluid parcels (i.e., points that move with the fluid velocity field, 'lumps' of the continuum but not necessarily or probably molecules)

which never leave the interface. Consider one such parcel. It moves vertically (i) if the interface rises or falls, or (ii) if the fluid flows horizontally under the sloping interface. If we let $z = \eta(x, y, t)$ be the interface, then

$$w[x, y, \eta(x, y, t), t] = \eta_t + u\eta_x + v\eta_y \quad \text{at} \quad z = \eta$$

This is really just a restatement of $D\eta/Dt = w$. In terms of $\phi$, this says

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \quad \text{at} \quad z = \eta$$

This is nothing more than a *kinematic* condition which simply says what we mean by calling $z = \eta$ an interface.

The interface is massless. In the absence of surface tension, therefore, it supports no pressure differences across it. The appropriate *dynamical* boundary condition is

$$p(x, y, \eta, t) = p_{atmosphere}$$

To write this in terms of $\phi, \eta$ return to

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p/\rho - g\hat{k}$$

Using the identity

$$(\vec{u} \cdot \nabla)\vec{u} = (\nabla \times \vec{u}) \times \vec{u} + \nabla(\vec{u} \cdot \vec{u}/2)$$

we can rewrite this (exactly) as

$$\vec{u}_t + \vec{\omega} \times \vec{u} = -\nabla p/\rho - \nabla(\vec{u} \cdot \vec{u}/2) - \nabla gz$$

Now if $\vec{\omega} = 0$ so that $\vec{u} = \nabla\phi$, then this becomes

$$\nabla(\phi_t + p/\rho + \frac{1}{2}|\nabla\phi|^2 + gz) = 0$$

$$\phi_t + p/\rho + gz + \frac{1}{2}|\nabla\phi|^2 = f(t)$$

which is the Bernoulli integral. We apply this at $z = \eta$ to find

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = f(t) - p_{atm}/\rho$$

The function $f(t)$ may be chosen to cancel the space independent part of $p_{atm}(x, y, t)$. We may as well do this since $f(t)$ only adds a space independent part to $\phi$. For constant (i.e., spatially non-varying) $p_{atm}$, we then have

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \quad \text{at} \quad z = \eta$$

Notice how a specified $p_{atm}(x, y, t)$ would enter the problem through this boundary condition.

The full problem is

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \quad \text{at} \quad z = \eta$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \quad \text{at} \quad z = \eta$$

$$\nabla^2\phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

## 3.2 Linear solutions

To get some idea of possible solutions, we will linearize and solve in one horizontal dimension. For now we just drop the nonlinear terms. We will check *a posteriori* that they are small compared with the linear terms. The linearized problem is

$$\eta_t = \phi_z \quad \text{at} \quad z = 0$$

$$\phi_t + g\eta = 0 \quad \text{at} \quad z = 0$$

$$\nabla^2\phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

Notice that the surface conditions have been applied at $z = 0$. We seek plane wave solutions $\eta = ae^{-i\sigma t + ikx}$ and $\phi = Ae^{-i\sigma t+ikx}Z(z)$. The interior equation gives $-k^2Z + Z_{zz} = 0$ which has the solutions $Z(z) = e^{\pm kz}$. The linear combination of these that satisfies the bottom boundary condition is $Z(z) = \cosh k(z + D)$. The free surface conditions may be combined into

$$\phi_{tt} + g\phi_z = 0 \quad \text{at} \quad z = 0$$

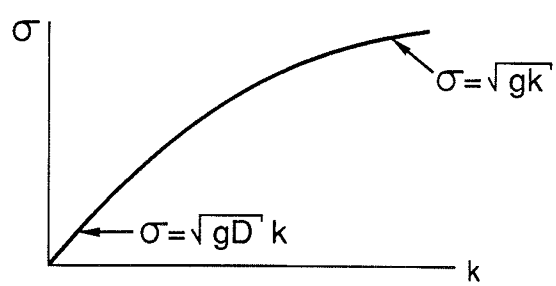
The solution

$$\phi = Ae^{-i\sigma t+ikx}\cosh k(z + D)$$

satisfies this provided

$$\sigma^2 = gk \tanh kD$$

which is the dispersion relation.




Finally $\eta_t = \phi_z$ and $\phi_t + g\eta = 0$ say that if $\eta = ae^{-i\sigma t+ikx}$, then

$$A = -ia\sigma/(k \sinh kD) = -iag/(\sigma \cosh kD)$$

These are the plane wave solutions. They are dispersive and the same wavelength can propagate in either the $+x$ or the $-x$ direction.

For completeness, we take the real parts

$$\begin{aligned}
\eta &= a\cos(kx-\sigma t)\\
\phi &= \frac{a\sigma}{k\sinh kD}\cosh k(z+D)\sin(kx-\sigma t)\\
u &= \phi_x = \frac{a\sigma}{\sinh kD}\cosh k(z+D)\cos(kx-\sigma t)\\
w &= \phi_z = \frac{a\sigma}{\sinh kD}\sinh k(z+D)\sin(kx-\sigma t)\\
p &= -\rho g z + \frac{\rho\sigma^2 a}{k\sinh kD}\cosh k(z+D)\cos(kx-\sigma t)\\
\sigma^2 &= gk\tanh kD
\end{aligned}$$

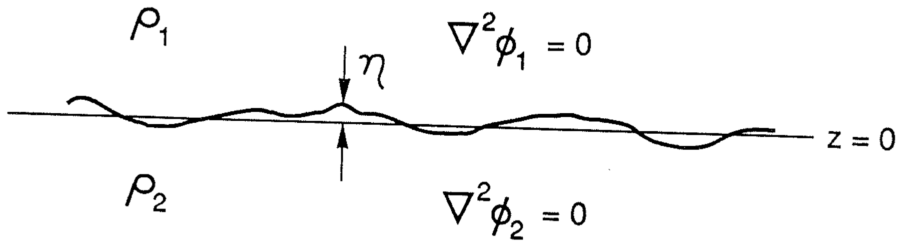
Notice that $\eta, u, p$ are in phase and that $p$ is not hydrostatic.

The above derivation is valid for waves of any wavelength and for fluid of any depth. However, the case of *deep water* waves for which the depth of the fluid is much greater than the wavelength of the wave, $kD \to \infty$, may be more appropriate to some waves in the deep ocean. In this limit, the plane wave solutions become

$$\begin{aligned}
\eta &= a\cos(kx-\sigma t)\\
\phi &= \frac{a\sigma}{k}e^{kz}\sin(kx-\sigma t)\\
\sigma^2 &= gk
\end{aligned}$$

## 3.3 Internal waves

The interface between the atmosphere and a body of water is not the only interface which can support gravity waves. In fact, any interface separating two fluids can support gravity waves. Consider the interface between two semi-infinite fluids of different densities. We have




At the interface $z = 0$

$$\eta_t = \phi_{1z} \quad ; \quad \eta_t = \phi_{2z}$$

$$\rho_1(\phi_{1t} + g\eta) = \rho_2(\phi_{2t} + g\eta)$$

We can satisfy these equations and the finiteness of the solution as $z \to \pm\infty$ by taking

$$\begin{aligned}\phi_1 &= A_1 e^{-i\sigma t + ikx - kz} \\ \phi_2 &= A_2 e^{-i\sigma t + ikx + kz} \\ \eta &= a e^{-i\sigma t + ikx}\end{aligned}$$

The three interface conditions become

$$-i\sigma a = -kA_1 \quad ; \quad -i\sigma a = kA_2 \quad ; \quad \rho_1(-i\sigma A_1 + ga) = \rho_2(-i\sigma A_2 + ga)$$

which yields

$$A_1 = ia\sigma/k \quad ; \quad A_2 = -ia\sigma/k \quad ; \quad \rho_1(\sigma^2/k + g) = \rho_2(-\sigma^2/k + g)$$

The latter may be rewritten

$$\sigma^2 = gk\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$$

Note that if $\rho_1 = 0$, then we recover the deep water dispersion relation of the previous section, $\sigma^2 = gk$.

For general $\rho_1 < \rho_2$, the quantity $g(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ can be regarded as a *reduced gravity*, typically denoted by $g'$. In the ocean, $g' \sim O(10^{-3})g$. These interfacial waves are called *internal waves*, and from $\sigma^2 = g'k$ we see that they move much more slowly than surface waves. We will spend several future lectures examining internal waves in much greater detail.

Note also that if $\rho_1 > \rho_2$, then $\sigma^2 < 0$ so that $\sigma$ is imaginary. Now $e^{-i\sigma t}$ represents exponential growth or decay in time. This corresponds to gravitational instability of the interface because heavier fluid overlays lighter fluid.

## 3.4 Qualitative retreatment of surface waves

Let's redo the problem of surface gravity waves to bring out a few points.

a) The full momentum equations are $D\vec{u}/Dt = -\nabla p^*/\rho - g\hat{k}$. Separate $p^*$ as $p^* = p_0(z) + p(\vec{x}, t)$ where $p_0$ is the hydrostatic part of the pressure which satisfies $0 = -p_{0z}/\rho - g$ and $p$ is a small perturbation from $p_0$. The linearized momentum equations become (in one horizontal dimension)

$$u_t = -p_x/\rho \quad ; \quad w_t = -p_z/\rho \quad ; \quad u_x + w_z = 0$$

At the bottom $w = 0$, i.e., $p_z = 0$ at $z = -D$. At the surface, $Dp^*/Dt = 0$ at $z = \eta$, for which the linearization is $p_t + wp_{0z} = 0$ at $z = 0$. Using the definition for the hydrostatic pressure, this becomes $p_t - g\rho w = 0$ at $z = 0$, or using the vertical momentum equation, $p_{tt} + gp_z = 0$ at $z = 0$. Now compare these with the results of the previous linearization:

$$
\begin{array}{cc}
u_t = -p_x/\rho & u = \phi_x \\
w_t = -p_z/\rho & w = \phi_z \\
u_x + w_z = 0 \Rightarrow \nabla^2 p = 0 & \nabla^2\phi = 0 \\
p_{tt} + g p_z = 0 \text{ at } z = 0 & \phi_{tt} + g\phi_z = 0 \text{ at } z = 0 \\
p_z = 0 \text{ at } z = -D & \phi_z = 0 \text{ at } z = -D
\end{array}
$$

We see that, in this linearized problem, $p = -\rho\phi_t$ which could also have been obtained from the Bernoulli equation.

b) When is the linearization valid? To answer this, consider the surface condition $\phi_t + g\eta + \frac{1}{2}|\nabla\phi|^2 = 0$ at $z = \eta$. The linearization is $\phi_t + g\eta = 0$ at $z = 0$. Now

$$
\begin{aligned}
\phi_t|_{z=\eta} &= \phi_t|_{z=0} + \eta\phi_{tz}|_{z=0}\cdots \\
&= [-i\sigma A + a(-i\sigma kA)e^{-i\sigma t+ikx}]e^{-i\sigma t+ikx}\cdots
\end{aligned}
$$

where we have used $\phi = Ae^{-i\sigma t+ikx+kz}$ which is appropriate for deep water waves. So we see that $\eta\phi_{tz} \ll \phi_t$ provided $-i\sigma kAa \ll -i\sigma A$. That is, provided that $ak \ll 1$. This means that the linearization is valid for waves which have a gentle slope. Evidently deep water waves are the beginning of an expansion in $(ak)$ of solutions to the full equations. We will return to a more formal expansion of the equations shortly.

c) The foregoing linearization yielded

$$
u_t = -p_x^*/\rho \quad ; \quad w_t = -p_z^*/\rho - g \quad ; \quad u_x + w_z = 0
$$

Suppose the wavelength $\lambda$ of the wave is much longer than the water depth $D$. Then $u_x + w_z = 0$ becomes, in order of magnitude, $u/\lambda = w/D$ or $w = uD/\lambda$. If $D/\lambda \to 0$, then $w \to 0$ and the pressure becomes entirely hydrostatic, $0 = -p_z^*/\rho - g$. Hence $p^* = g\rho(\eta - z)$ which leads to the new linearized momentum equation of

$$
u_t = -g\eta_x
$$

Notice that this implies $u$ is a function of $x$ and $t$ but not a function of $z$ since $\eta = \eta(x,t)$.

Now, from continuity and $u \neq u(z)$,

$$\begin{aligned} \int_{-D}^{\eta} (w_z + u_x)\, dz &= 0 \\ \eta_t + u\eta_x + \int_{-D}^{\eta} u_x\, dz &= 0 \\ \eta_t + [u(\eta + D)]_x &= 0\,. \end{aligned}$$

Linearization of this ($\eta \ll D$) yields

$$\eta_t + Du_x = 0$$

which, along with the new momentum equation above, are the *linearized shallow water equations*, so called because $D \ll \lambda$. Eliminating $u$ between them yields

$$\eta_{tt} - gD\eta_{xx} = 0$$

which is simply a one-dimensional wave equation with $c = (gD)^{1/2}$. From this, if $\eta = ae^{-i\sigma t + ikx}$, then $\sigma/k = \pm(gD)^{1/2}$ and

$$\begin{aligned} u &= \frac{gak}{\sigma} e^{-i\sigma t + ikx} \\ w &= \frac{-igak^2}{\sigma}(z + D)e^{-i\sigma t + ikx} \\ p^* &= g\rho(\eta - z) \end{aligned}$$

Note that $w \neq 0$, but rather $w \ll u$, and we will see shortly that $w$ enters the solution at second order.

## 3.5 Careful retreatment of surface waves

The last sections have shown that the waves are very different depending on whether the wavelength is much greater or much less than the depth. A more systematic treatment returns to the full problem

$$\phi_t + g\eta + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \quad \text{at} \quad z = \eta$$

$$\eta_t + (\phi_x \eta_x + \phi_y \eta_y) = \phi_z \quad \text{at} \quad z = \eta$$

$$\nabla^2 \phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -D$$

Introduce the following scaling

$$\begin{aligned}
\text{dimensional} &= \text{dimensionless} \\
(x, y) &= (x, y)L \\
z &= zD \\
t &= tL/(gD)^{1/2} \\
\eta &= \eta a \\
\phi &= \phi \frac{gaL}{(gD)^{1/2}} \quad (\text{from} \quad \phi_t + g\eta \simeq 0)
\end{aligned}$$

For example now

$$\begin{aligned}
\eta_t + \phi_x \eta_x &= \phi_z \\
\text{becomes} \quad \frac{a(gD)^{1/2}}{L}\eta_t + \frac{gaL}{(gD)^{1/2}} \frac{a}{L^2}\phi_x \eta_x &= \frac{gaL}{D(gD)^{1/2}}\phi_z \\
\eta_t + (a/D)\phi_x \eta_x &= (L/D)^2 \phi_z \quad \text{at} \quad z = (a/D)\eta
\end{aligned}$$

We see that the only two dimensionless numbers that appear are

$$\epsilon \equiv a/D \quad ; \quad \delta \equiv D/L$$

which are called the amplitude and the aspect ratio, respectively.

We obtain for all of the equations:

$$\phi_t + \frac{\epsilon}{2}(\phi_x^2 + \phi_y^2) + \frac{\epsilon\delta^{-2}}{2}\phi_z^2 + \eta = 0 \quad \text{at} \quad z = \epsilon\eta$$

$$\eta_t + \epsilon(\eta_x\phi_x + \phi_y\eta_y) = \delta^{-2}\phi_z \quad \text{at} \quad z = \epsilon\eta$$

$$\phi_{zz} + \delta^2(\phi_{xx} + \phi_{yy}) = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -1$$

a) Notice that if we take $\epsilon \ll 1,\ \delta = 1$, then we obtain

$$\phi_t + \eta = 0 \quad ; \quad \eta_t = \phi_z \quad \text{at} \quad z = 0$$

$$\nabla^2\phi = 0$$

$$\phi_z = 0 \quad \text{at} \quad z = -1$$

which is the dimensionless version of the deep water wave problem we previously derived.

b) Consider now $\epsilon = 1,\ \delta \ll 1$. Let $\phi = \phi_0 + \delta^2\phi_2 + ...$, and insert this into the interior equation. At lowest order

$$\delta^{(0)}: \quad \phi_{0zz} = 0 \quad ; \quad \phi_{0z} = 0 \quad \text{at} \quad z = -1$$

which means that $\phi_0$ is independent of $z$. At next order

$$\delta^{(2)}: \quad \phi_{2zz} + \phi_{0xx} + \phi_{0yy} = 0 \quad ; \quad \phi_{2z} = 0 \quad \text{at} \quad z = -1$$

which yields

$$\phi_2(x, y, z, t) = -(z+1)^2(\phi_{0xx} + \phi_{0yy})/2$$

Therefore

$$\phi = \phi_0(x, y, t) - (z+1)^2\delta^2(\phi_{0xx} + \phi_{0yy})/2$$

This expression is now substituted into the surface boundary conditions to obtain

$$\phi_{0t} + \frac{1}{2}(\phi_{0x}^2 + \phi_{0y}^2) + O(\delta^2) + \eta = 0$$

$$h_t + (h\phi_{0x})_x + (h\phi_{0y})_y = 0 + O(\delta^2)$$

where $h = 1 + \eta$. Note that these equations are now the field equations because the dependence on $z$ is gone. Recall that $u_0 = \phi_{0x}$, $v_0 = \phi_{0y}$, etc. Inserting these into the above yields

$$\begin{aligned} h_t + (u_0 h)_x + (v_0 h)_y &= 0 \\ u_{0t} + u_0 u_{0x} + v_0 u_{0y} &= -\eta_x \\ v_{0t} + u_0 v_{0x} + v_0 v_{0y} &= -\eta_y \end{aligned}$$

These equations can be converted to the dimensional form by multiplying the right hand side of the second and third equations by $g$ and by interpreting $h$ as $D + \eta$. These are the *nonlinear shallow water equations.* We previously arrived at their linearization by heuristic reasoning. Note that $w = \phi_z = \delta^2 \phi_{2z}$ which is $O(\delta^2)$, in agreement with the result that we obtained using heuristic arguments.

## 3.6 An initial value problem

Now that we have established appropriate linearized equations for deep and shallow water waves, we consider an application. Since $\sigma^2 = gk$ is quadratic, it is likely that solutions for $\eta(x,t)$ on $-\infty < x < \infty$ require specification of $\eta(x,0)$ and $\eta_t(x,0)$. If we set

$$\eta(x,t) = \int_{-\infty}^{\infty} [C(k)e^{i\sigma t + ikx} + D(k)e^{-i\sigma t + ikx}]\, dk$$

then

$$\begin{aligned} \eta(x,0) &= \int_{-\infty}^{\infty} [C(k) + D(k)]e^{ikx}\, dk \\ \eta_t(x,0) &= \int_{-\infty}^{\infty} i\sigma[C(k) - D(k)]e^{ikx}\, dk \end{aligned}$$

For simplicity, consider $\eta_t(x,0) = 0$. Then $2C(k) = \bar{\eta}_0$ and

$$\eta(x,t) = 1/2 \int_{-\infty}^{\infty} \bar{\eta}_0(k)(e^{i\sigma t+ikx} + e^{-i\sigma t+ikx})\, dk$$

where $\sigma = (\text{sign}k)\ (g|k|)^{1/2}$ to ensure that waves at frequency $\sigma$ propagate in the same direction regardless of the sign of $k$. The solution may be separated into left and right going wave contributions by writing

$$\eta(x,t) = 1/2 \int_{-\infty}^{\infty} \bar{\eta}_0(k)e^{i\Theta_+ t}\, dk + 1/2 \int_{-\infty}^{\infty} \bar{\eta}_0(k)e^{i\Theta_- t}\, dk$$

where

$$\Theta_\pm \equiv kx/t \pm (\text{sign}k)\ (g|k|)^{1/2}$$

Points of stationary phase are where $\Theta'_\pm = 0$ (the prime means $\partial/\partial k$). Now $\Theta'_+ = 0$ has no real root for $x > 0$, so we must use $\Theta_-$

$$\eta(x > 0, t) = 1/2 \int_{-\infty}^{\infty} \bar{\eta}_0(k)e^{i\Theta_- t}\, dk$$

Thus, for $k > 0$

$$\Theta_- = kx/t - (gk)^{1/2}$$

whence

$$\Theta'_- = x/t - \frac{1}{2}(g/k)^{1/2} = 0$$

yields

$$x/t = \frac{1}{2}(g/k_0)^{1/2}$$

and

$$\Theta''_-(k_0) = \frac{g^{1/2}}{4k_0^{3/2}} = \frac{2x^3}{gt^3}$$

Thus

$$\eta_{k>0}(x > 0, t) \simeq \frac{1}{2}\bar{\eta}_0(k_0)e^{i\Theta_-(k_0)t}[2\pi/it\Theta''_-(k_0)]^{1/2}$$

becomes

$$\eta_{k>0}(x>0,t)=\frac{1}{2}\bar{\eta}_0(k_0)e^{-igt^2/4x}e^{-i\pi/4}(\pi g t^2/x^3)^{1/2}$$

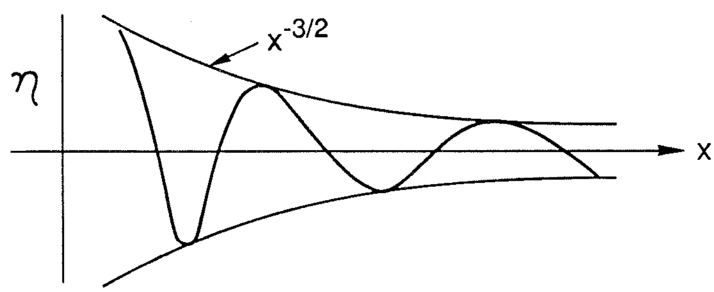
An identical contribution obtains from $k<0$, so

$$\eta(x>0,t)=\bar{\eta}_0(k_0)(\pi g)^{1/2}\frac{t}{x^{3/2}}\cos(gt^2/4x+\pi/4)$$

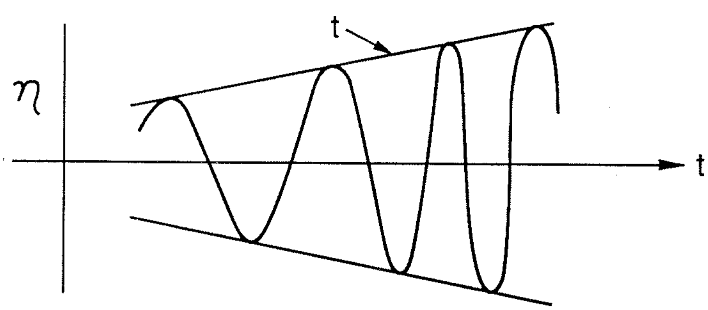
If $\eta_0(x,0)=\delta(x)$, then $\bar{\eta}_0(k_0)=1/2\pi$ and then

$$\eta(x>0,t)=\frac{1}{2}(g/\pi)^{1/2}\frac{t}{x^{3/2}}\cos(gt^2/4x+\pi/4)$$

If we plot $\eta(x,t)$ versus $x$ at a fixed $t$ (snapshot), we see




The wavelength $2\pi/k_0$ increases and the amplitude decreases with $x$. A wavestaff record of $\eta(x,t)$ at fixed $x$ shows




Clearly neither $k_0$ nor $\sigma_0$ is constant at any fixed $x$ or $t$. Yet it turns out that

$$\sigma_{0t}+\frac{\partial\sigma}{\partial k}|_{k_0}\sigma_{0x}=0 \quad ; \quad k_{0t}+\frac{\partial\sigma}{\partial k}|_{k_0}k_{0x}=0$$

i.e., if we travel at $x/t=\partial\sigma/\partial k|_{k_0}$, then $\sigma_0$ and $k_0$ do not change.

It is instructive to consider the same problem from the ray theory point of view. Ray theory postulates a solution of the form $\eta=ae^{iP}$, defines a local wavenumber $k$ by

$k = \partial P/\partial x|_t$ and a local frequency $N$ by $N = -\partial P/\partial t|_x$, and asserts that these satisfy the plane wave dispersion relation $N = \Omega(k)$. We now see that the stationary phase solution does all of these things as well. Since $P = t\Theta = k_0 x - \Omega(k_0)t$, we have

$$N = \Omega(k_0) + (x - \frac{\partial\Omega}{\partial k_0}t)\frac{\partial k_0}{\partial t} = \Omega(k_0) = \sigma_0$$

$$k = k_0 + (x - \frac{\partial\Omega}{\partial k_0}t)\frac{\partial k_0}{\partial x} = k_0$$

where $\sigma_0 = \Omega(k_0)$ and $x/t = \partial\Omega/\partial k_0$ have been used. Thus, the phase $P$ and the local wave parameters $k_0$, $\sigma_0$ satisfy the relationships asserted by ray theory. We further have

$$-\frac{\partial}{\partial x}\frac{\partial P}{\partial t}|_x + \frac{\partial}{\partial t}\frac{\partial P}{\partial x}|_t = 0$$

i.e., $\sigma_{0x} + k_{0t} = 0$. Since $\sigma_0 = \Omega(k_0)$ and $\sigma_{0t} = (\partial\Omega/\partial k_0)k_{0t}$, then we recover

$$\sigma_{0t} + \frac{\partial\sigma}{\partial k}|_{k_0}\sigma_{0x} = 0 \;\; ; \;\; k_{0t} + \frac{\partial\sigma}{\partial k}|_{k_0}k_{0x} = 0$$

which again says that $k_0$ and $\sigma_0$ do not change if we travel at the group velocity. This much all by itself tells us that if we are at $(x,t)$, then the solution there looks like $ae^{-i\sigma_0 t + ik_0 x}$ where $\partial\sigma/\partial k|_{k_0} = x/t$.

Ray theory also tells us how to get the amplitude which, in this case, is prescribed by

$$\varepsilon_t + (c_g\varepsilon)_x = 0$$

where $\varepsilon = \frac{1}{2}\rho g\eta\eta^*$. So, the stationary phase approximation to this initial value problem in a homogeneous dispersive medium could have been obtained by the simpler ray theory approach.

At any $(x,t)$, the stationary phase solution has well defined frequency $\sigma_0$ and wavenumber $k_0$. This is because, at the long times $t$ for which the stationary phase approximation is valid, dispersion has separated the concentrated initial disturbance into a slowly varying wave train of the sort postulated beforehand by the ray theory. Neither theory handles the details of how the solution evolves near the initial disturbance.

Note that as $t \to \infty$, the foregoing says $\eta(x, t \to \infty) = t$. The solution never 'settles down'. This happens because $\eta_0(x, 0) = \delta(x)$ contains infinitely short waves that travel infinitely slowly. Therefore, at any given $(x, t)$, short waves are still arriving and shorter ones are en route. This does not happen for the finite initial displacement.

## 3.7 Ship waves

Let's consider another application. We have seen, in one dimension, that $\eta_0(x, 0) = \delta(x)$ , $\eta_{0t}(x, 0) = 0$ leads to

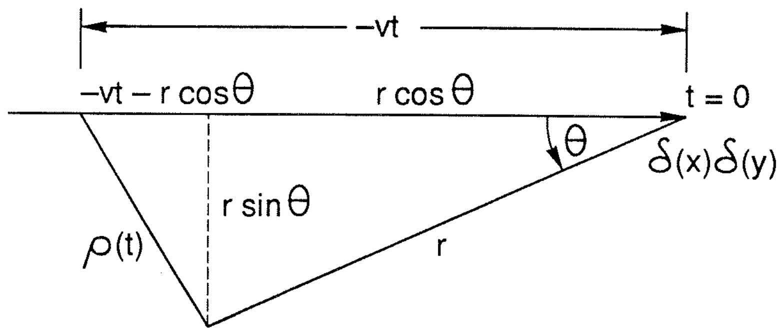
$$\eta(x, t) = \Re K_1 \frac{P^{1/2}}{x} e^{iP}$$

where $P = gt^2/4x$. In two dimensions, i.e. radial spreading, it turns out that $\eta_0(x, y, 0) = \delta(x)\delta(y)$ , $\eta_{0t}(x, y, 0) = 0$ leads to

$$\eta(x, y, t) = \Re K_2 \frac{P}{r^2} e^{iP}$$

where $P = gt^2/4r$ and $r = (x^2 + y^2)^{1/2}$. The dispersive characteristics of the wave train – summarized in $e^{iP}$ – are common to both one dimension and two dimensions although the envelope changes from one dimension to two dimensions.

We use the two dimensional result to discuss ship waves. A ship is idealized as a travelling delta function which moves with speed $V$.




At time $t = 0$, we are at $r, \theta$ relative to the ship. At time $t$ the ship was $\rho(t)$ from us. Keep in mind that $t < 0$. We have, therefore,

$$\eta(r, \theta, t = 0) = \int_{-\infty}^{0} K_2 \frac{P}{\rho^2(t)} e^{iP(t)}\, dt$$

where $P(t) = gt^2/4\rho(t)$ and $\rho(t) = (r^2 + V^2t^2 + 2Vtr\cos\theta)^{1/2}$.

This is like a stationary phase problem if $P(t)$ is large. Points of stationary phase are when $P_t = 0$, i.e.

$$\frac{2gt}{4\rho} - \frac{gt^2\rho_t}{4\rho^2} = 0$$

$$\frac{2gt}{4\rho} - \frac{gt^2}{4\rho^2}\frac{1}{2\rho}(2V^2t + 2Vr\cos\theta) = 0$$

$$2\rho^2 - V^2t^2 - Vrt\cos\theta = 0$$

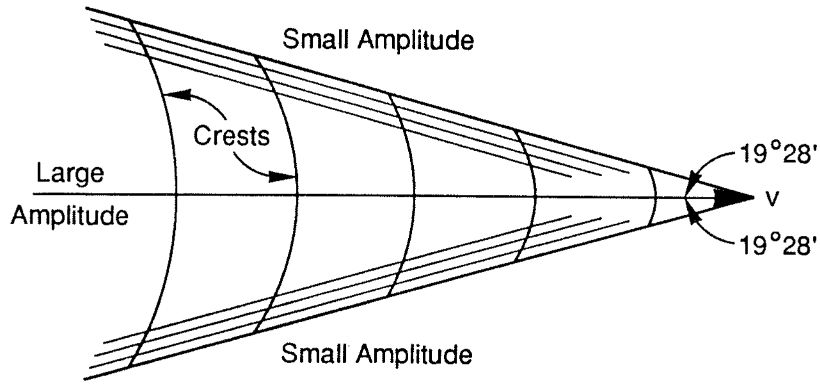
$$2(r^2 + V^2t^2 + 2Vrt\cos\theta) - V^2t^2 - Vrt\cos\theta = 0$$

$$2r^2 + V^2t^2 + 3Vrt\cos\theta = 0$$

$$t_{\pm} = -\frac{3r}{2V}[\cos\theta \pm (\cos^2\theta - 8/9)^{1/2}]$$

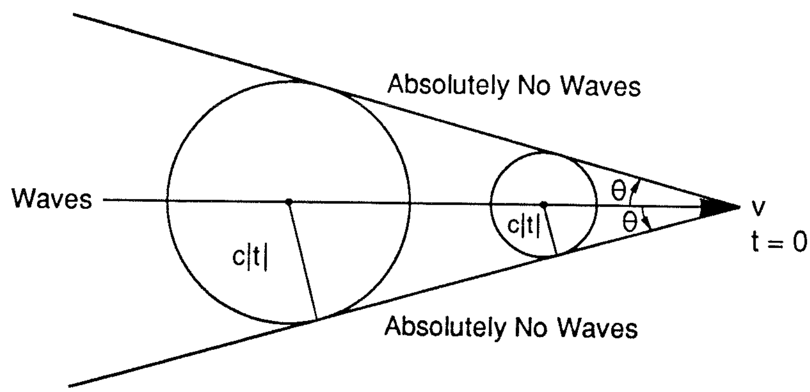
We get an appreciable contribution only when $t_{\pm}$ lie on the path of integration $-\infty$ to 0, i.e. when they are real. This requires $\cos^2\theta > 8/9$, i.e. $\theta < 19°28'$. So, for $\theta > 19°28'$ we get far smaller waves than for $\theta < 19°28'$. Notice that this angle is independent of $V$. This means that the waves following a ship will be at the same angle regardless of the speed that the ship travels! (Of course, the ship must be idealized as a point source.)

Inside this cone there are two wave systems $e^{iP(t_+)}$ and $e^{iP(t_-)}$. They give rise to the system of cross waves seen behind a ship.




Details of their shape come from $P(t_+) = constant$ and $P(t_-) = constant$.

The $V$ independence is surprising. But remember that these waves are dispersive – some always travel as fast as the ship regardless of its speed. The nondispersive case is different. If all waves travel at $c$ [$= (gD)^{1/2}$ in a shallow sea] and a ship moves at $V > c$, then the wave pattern looks like




The waves are confined to $\theta < \sin^{-1}(ct/Vt)$ which *is* dependent on the velocity just as we would expect. This is because the waves all travel slower than the ship. The waves arrive as a sharp discontinuity.

## 3.8 A wave energy equation

The linearized waves satisfy

$$\rho\vec{u}_t = -\nabla p - g\rho\hat{k}$$

$$\nabla\cdot\vec{u} = 0$$

From these

$$(\frac{1}{2}\rho\vec{u}\cdot\vec{u})_t + \vec{u}\cdot\nabla p + g\rho w = 0$$

In the linearized case, $w = z_t$ which can be used with continuity to obtain

$$[\frac{1}{2}\rho\vec{u}\cdot\vec{u} + g\rho z]_t + \nabla\cdot\vec{u}p = 0$$

$$[ke + pe]_t + \nabla\cdot\ eflux = 0$$

Integrate from $z = -D(x)$ to $z = \eta$ and note that

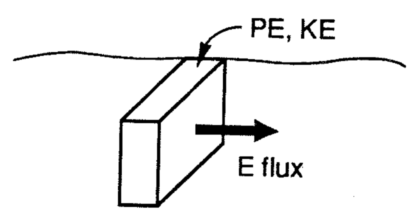
$$\begin{aligned}\int_{-D}^{\eta}[\partial_x(up) + \partial_y(vp) + \partial_z(wp)]\,dz &= \partial_x\int_{-D}^{\eta} up\,dz - p(\eta)u\eta_x + p(-D)uD_x\ldots \\ &\quad +wp(\eta) - wp(-D) \\ &= \partial_x\int_{-D}^{\eta} up\,dz\end{aligned}$$

Thus

$$[\int_{-D}^{\eta}\frac{1}{2}\overline{\rho\vec{u}\cdot\vec{u}}\,dz + \frac{1}{2}\rho g\overline{\eta^2}]_t + \nabla_H\cdot\int_{-D}^{\eta}\overline{\vec{u}p}\,dz = 0$$

$$[\overline{KE} + \overline{PE}]_t + \nabla_H\cdot\overline{Eflux} = 0$$

where $KE$ and $PE$ are energy densities per unit surface area, $\nabla_H = \hat{\imath}\partial/\partial x + \hat{\jmath}\partial/\partial y$ and the overbar denotes a time average over one wave period.




For

$$\begin{aligned}\eta &= a\cos(kx - \sigma t) \qquad \sigma^2 = gk\tanh kD \\ \phi &= \frac{a\sigma}{k\sinh kD}\cosh k(z + D)\sin(kx - \sigma t) \\ p &= -g\rho z + \frac{\rho\sigma^2 a}{k\sinh kD}\cosh k(z + D)\cos(kx - \sigma t)\end{aligned}$$

we find that

$$\overline{KE} = \overline{PE} = \frac{1}{4}\rho g a^2$$

so that the total energy is $\overline{E} = \frac{1}{2}\rho g a^2$. Finally, after some algebra, the energy flux can be written

$$\begin{aligned} \overline{Eflux} &= \int_{-D}^{0} \overline{up}\, dz \\ &= \underbrace{\frac{1}{2}\rho g a^2}_{\overline{E}} \underbrace{(\frac{\sigma^2}{gk}\coth kD)}_{1} \underbrace{\frac{\sigma}{2k}(1 + 2kD/\sinh 2kD)}_{\partial\sigma/\partial k} \end{aligned}$$

That is

$$\overline{Eflux} = \overline{E}\vec{c}_g$$

Thus, the period average of the energy equation is, for the plane wave

$$\overline{E}_t + \nabla_H \cdot (\overline{E}\vec{c}_g) = 0$$

It may be used to determine $\overline{E}(\vec{x}, t)$ from $\overline{E}(\vec{x}, 0)$ if the wave is slowly varying, i.e. if $a = a(\vec{x}, t)$. This may occur either if $a(\vec{x}, 0)$ is slowly varying or if $D$ is a slowly varying function of position.

## 3.9 Slowly varying medium

The 'medium' is made nonuniform if the fluid depth is variable in space or (rarely) in time. The techniques used up to now accomodate this case with little further thought. However, medium nonuniformity also occurs if the waves advance through currents. If the currents vary only slightly over a wave period or wavelength, then the waves may be adequately described by slowly varying representation.

For concreteness, consider a basic flow $U(x,y,t)$, $V(x,y,t)$, $W(x,y,z,t)$, $P(x,y,z,t)$ with a free surface $z = h(x,y,t)$ flowing over relief $z = -D(x,y,t)$. It satisfies

$$U_t + UU_x + VU_y = -P_x/\rho$$

$$V_t + UV_x + VV_y = -P_y/\rho$$

$$U_x + V_y + W_z = 0 \quad \text{or} \quad (h+D)_t + [U(h+D)]_x + [V(h+D)]_y = 0$$

Since it is to be slowly varying in the sense that $\epsilon = L_w/L_m \ll 1$, then we require $h_x$, $D_x$ etc. to be $O(\epsilon)$. This means that $W$ is $O(\epsilon)U$. The pressure is hydrostatic, i.e. $P = \rho g(h - z)$.

Now let $u^* = U + u$, $v^* = V + v$, $w^* = W + w$, $p^* = P + p$, $\eta^* = h + \eta$. We have, for example,

$$\frac{Du^*}{Dt} = -p^*_x/\rho \rightarrow (U+u)_t + UU_x + Uu_x + uU_x + uu_x \ldots = -P_x/\rho - p_x/\rho$$

Using the definitions of $U, V, W, P$ and linearizing by neglecting products of small terms yields

$$u_t + Uu_x + uU_x + Vu_y + vU_y = -p_x/\rho$$

and similar equations for $u$ and $v$. Making the further assumption that $U$ and $V$ are slowly varying results in

$$\begin{aligned} u_t + Uu_x + Vu_y &= -p_x/\rho \\ v_t + Uv_x + Vv_y &= -p_y/\rho \\ w_t + Uw_x + Vw_y &= -p_z/\rho - g \end{aligned}$$

(Note that terms like $Wu_z$ are dropped because $W \sim Uh_x$ or $UD_x \sim \epsilon U$.)

At the free surface $Dp^*/Dt = 0$ at $z = \eta^*$. Using the same assumptions as above and noting that $P_t + UP_x + VP_y + WP_z = 0$, we arrive at

$$p_t + Up_x + Vp_y = g\rho w \quad \text{at} \quad z = 0$$

Finally $w^* = u^* D_x + v^* D_y$ at $z = -D$, which becomes $w = 0$ at $z = -D$. If we assume a plane wave solution $\eta = ae^{-i\sigma t + ikx + i\ell y}$ etc. then we obtain a dispersion relation of

$$\sigma = kU + \ell V + \left[g(k^2+\ell^2)^{1/2} \tanh D(k^2+\ell^2)^{1/2}\right]^{1/2}$$

which is simply that for surface gravity waves but Doppler shifted by the background current.

Using this dispersion relation, the ray theory recipe says that we can carry the slow space and time variation of $U$ and $D$ parametrically to find $N = \Omega(\vec{k}; x, y, t)$ or

$$\sigma = \vec{k} \cdot \vec{U}(x,y,t) + \left[g|\vec{k}| \tanh |\vec{k}| D(x,y,t)\right]^{1/2}$$

We may write this as $\sigma = \vec{k} \cdot \vec{U} + \sigma'$ where $\sigma' = (g|\vec{k}| \tanh |\vec{k}| D)^{1/2}$ is the frequency seen by an observer moving at $\vec{U}$. Then

$$\vec{c}_g = \frac{\partial \sigma}{\partial \vec{k}} = \vec{U} + \frac{\partial \sigma'}{\partial \vec{k}} = \vec{U} + \vec{c}_g'$$

Finally then, we find $\sigma(x,y,t), \vec{k}(x,y,t)$ by solving

$$\sigma_t + (\vec{U} + \vec{c}_g') \cdot \nabla \sigma = \Omega_t = \vec{k} \cdot \vec{U}_t + \frac{\partial}{\partial t}\left[g|\vec{k}| \tanh |\vec{k}| D(x,y,t)\right]^{1/2}$$

$$k_{it} + (\vec{U} + \vec{c}_g') \cdot \nabla k_i = -\Omega_{x_i} = -\vec{k} \cdot \frac{\partial \vec{U}}{\partial x_i} - \frac{\partial}{\partial x_i}\left[g|\vec{k}| \tanh |\vec{k}| D(x,y,t)\right]^{1/2}$$

These fix $\sigma(x,y,t), \vec{k}(x,y,t)$ once we are given $\sigma(x,y,0), \vec{k}(x,y,0)$. At least conceptually they are easy to integrate. To find the wave amplitude, we must formulate and solve an energy equation.

## 3.10 Waves riding on a current

We consider two examples which make use of the above formalism. First, let $D$ =constant and the current be $\vec{U} = \hat{\imath}U(x) \neq \hat{\imath}U(x,t)$. Now $\sigma = \Omega(k;x) = kU + (gk\tanh kD)^{1/2}$ from which

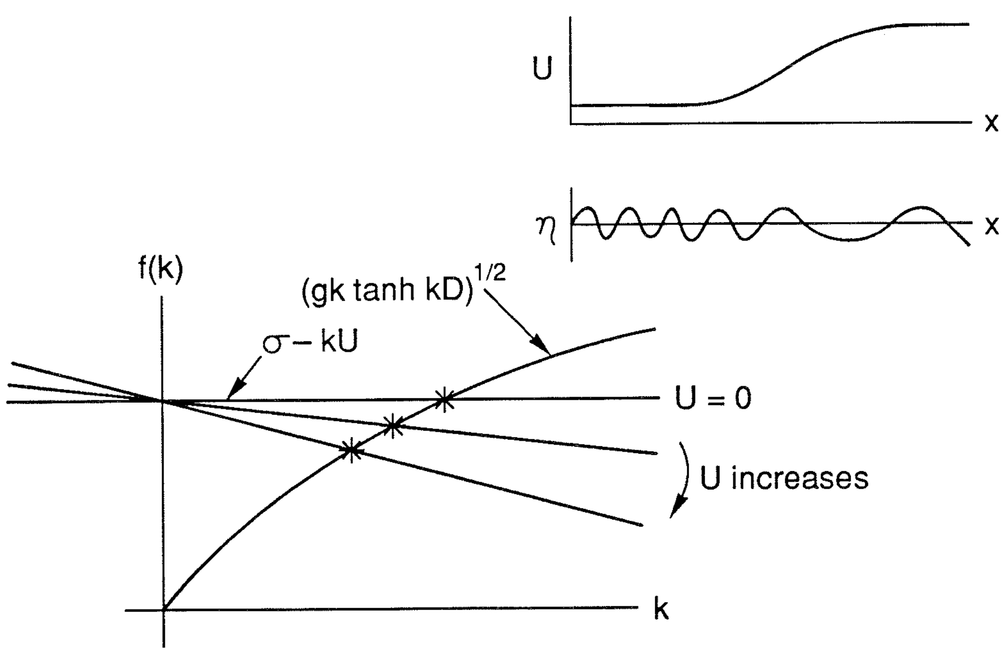
$$\sigma_t + (c_g' + U)\sigma_x = 0$$

If a wavemaker always puts waves of constant frequency $\sigma$ into the fluid at $x = 0$, this equation says that as we move at $c_g' + U$, $\sigma$ does not change. Ultimately this means that $\sigma$ is constant everywhere (but not $\sigma'$). Therefore

$$\sigma = k(x)U(x) + [gk(x)\tanh k(x)D]^{1/2}$$

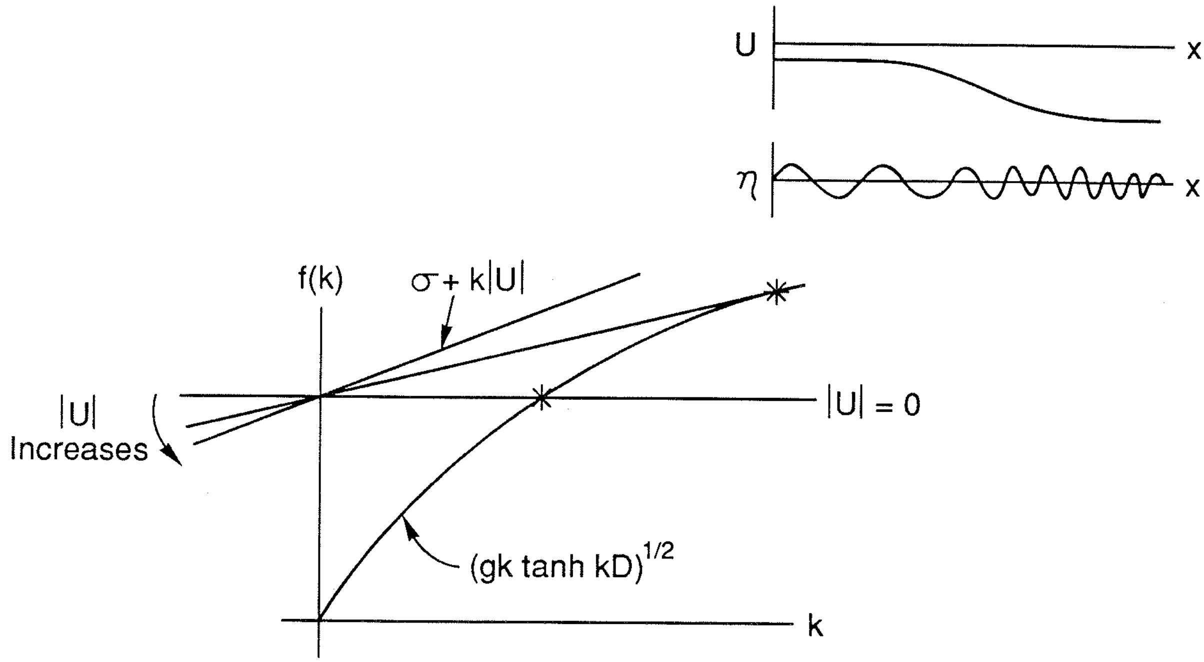
tells us $k(x)$, in principle.

Consider $\sigma, k > 0$ and $U(x) > 0$, i.e. right-going waves and current



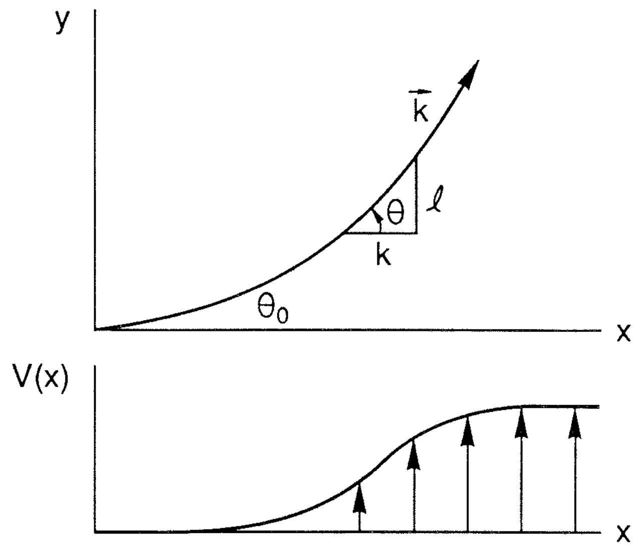

Clearly, there is always a root *. For large $U$, $\sigma \to kU$, i.e. $k \to \sigma/U$. The waves are longer in a swifter current.

Consider $\sigma, k > 0$ and $U(x) < 0$, i.e. right-going waves in a left-going current.




There is a root * for $0 < |U| < c_g'(k)$. At the upper limit $c_g'(k) = |U|$. Waves with smaller $k$ have larger $c_g'$ and can stem the current, while those with large $k$ go too slow to stem the current and are swept downstream. In reality, the waves break before this limit. (A second intersection of the two curves generally occurs at large $k$, but here $c_g' < |U|$, so such waves would never be realized.)

A second example is that of a shear flow $\vec{U} = \hat{\jmath}V(x)$. Waves started from a wavemaker at $x = 0$ at an angle $\theta_0$ to the $x$-direction refract as they pass through the current.




We have $\sigma = \ell V(x) + (gK \tanh KD)^{1/2}$ where $K^2 = k^2 + \ell^2$. As before, with $e^{-i\sigma t + ikx + i\ell y}$;

$$\sigma_t + \vec{c}_g \cdot \nabla \sigma = \Omega_t = 0$$

$$\ell_t + \vec{c}_g \cdot \nabla \ell = -\Omega_y = 0$$

$$k_t + \vec{c}_g \cdot \nabla k = -\Omega_x \neq 0$$

where $\sigma$ and $\ell$ are constant everywhere, but $k = k(x)$. The easiest way to find $k(x)$ is to realize that $\sigma = \ell V(x) + \left[g(k^2(x)+\ell^2)^{1/2} \tanh(k^2(x)+\ell^2)^{1/2} D\right]^{1/2}$ fixes $k(x)$. Now the relation $\ell = [k^2(x)+\ell^2]^{1/2} \sin\theta(x) = \text{constant}$ tells us $\theta(x)$.

For deep water, these are easy to solve:

$$\sigma = \ell V(x) + \left[g(k^2(x)+\ell^2)^{1/2}\right]^{1/2}$$

leads to

$$k^2(x) = \frac{[\sigma - \ell V(x)]^4}{g^2} - \ell^2$$

and

$$\frac{\sin\theta(x)}{\sin\theta_0} = \frac{(k_0^2+\ell^2)^{1/2}}{(k^2(x)+\ell^2)^{1/2}} = \frac{(\sigma - \ell V_0)^2}{(\sigma - \ell V(x))^2}$$

Notice that when $V(x) \to [\sigma - (g\ell)^{1/2}]/\ell$, then $k \to 0$ and the wave no longer propagates in the $x$-direction.